Power and Heat in Input Ouput Theory

A thesis presented for the degree of Bachelor of Science by

Max Schrauwen

Abstract

In quantum thermodynamics, open quantum systems can be described by Markovian quantum master equations. A different approach for describing open quantum systems that are probed with light is provided by input output theory. In this thesis we will compare two definitions for heat and work that are motivated by the two theoretical approaches, master equations and input output theory and quantify the differences between these two definitions. Furthermore we show that these definitions are both consistent with a set of thermodynamic laws for a simple harmonic oscillator model and pave the way towards a new framework of describing open quantum systems where the system's output is not entirely regarded as dissipation, but retains a coherent part, able to perform work.

September 17^{th} 2024

First Examiner: Prof. Dr. M. Berta, RWTH Aachen University
Second Examiner/Supervisor: Prof. Dr. P. Potts, University of Basel
Second Supervisor: A. Daniel, University of Basel

Department of Physics
University of Basel

Institute for Quantum Information
RWTH Aachen University

Contents

Introduction

One of the most fundamental questions in quantum thermodynamics deals with the decomposition of energetic changes into heat and work. Contrary to classical thermodynamics, heat and work do not have an unambiguous definition in quantum thermodynamics. Many different definitions may arise in the pursuit of decomposing the energetic changes of open quantum systems [1]. Nevertheless, heat and work remain fundamental quantities in the field of quantum thermodynamics as they describe the usefulness or the ability to perform work for any given system output. Quantifying the power and heat output of any system is therefore crucial in many fields of research concerning open quantum systems [2, 3].

However, as mentioned above, there is no unequivocal method of defining heat and work for open quantum systems and the absence of such a method may lead to confusion or incoherent results. One instance where this may be illustrated is in a case concerning input output theory, where the definition of work as the energy stored in the coherent part of the light as made by [4] is in contradiction to the commonly used definition in the framework of the quantum master equations. In light of this discrepancy, we express the existing definitions, consistent with the master equations, in terms of the central objects of input output theory and study the differences of this definition with the definition by [4].

We find that whereas in the conventional approach the output of the system consists mainly of dissipated heat, the new approach by [4] regards part of the output as work.

To shed light on the origin of this different viewpoint, we investigate the entropy production rate for both definitions and derive a second law of thermodynamics for both definitions for a simple system concerning a harmonic oscillator.

By exploring the definitions of heat and work we gain a deeper understanding of the fundamental nature of the energetic changes in open quantum systems. This helps to understand how to correctly quantify the ability to do work of any given system output and is therefore an essential extension to the use of input output theory for quantum thermodynamic purposes.

The rest of the thesis is structured as follows. In Sec. 1 we discuss the system and model for our calculations. We look at the different theories and the relevant equations and investigate the main observables in the steady state and the transient regime. In Sec. 2 we research and compare different definitions of the thermodynamic quantities power and heat for our system. We also verify the first law of thermodynamics for the relevant definitions. In Sec. 3 we investigate the entropy production rate and compare the different rates for the different definitions of power and heat. Here we verify the second law of thermodynamics for both definitions as well. Sec. 4 includes a conclusion of the thesis and covers the outlook for future work and application of the different definitions we discuss.

1 System and Model

1.1 Open Quantum Systems

The system of interest consist of a cavity driven by a coherent source. This may experimentally be achieved by using a laser driven cavity, where the cavity consists of a semi-transparent and a non-transparent mirror and the laser drive consist of coherent light with thermal noise. In the cavity a photon field can build up through the drive, which can be viewed as the input field. The cavity allows for dissipation as well as output in the form of coherent light, which can be viewed as the output field.
We can quantify the dissipation to the environment (which we will usually call the heat bath) through the coupling constant κ and the input and output fields through the averages of the input and output fields $\langle \hat{b}_{\text{in}} \rangle$ and $\langle \hat{b}_{\text{out}} \rangle$.
Fig. 1 presents an illustration of how an open quantum system could be represented.

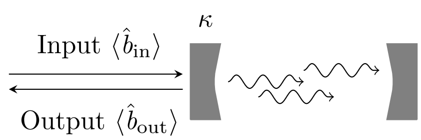


Figure 1: Illustration of an open quantum system with input and output field, dissipation and a cavity where a photon field can build up.

It is worth mentioning that the cavity will still contain photons even in the absence of a coherent drive. This is due to the thermal equilibrium described by the Bose-Einstein distribution

$$n_B = \frac{1}{e^{\epsilon/k_B T} - 1} \ . \tag{1}$$

Where ϵ is the energy, k_B the Boltzmann constant and T the temperature. This system can be described by a total Hamiltonian that consists of a system Hamiltonian, bath Hamiltonian and interaction Hamiltonian

$$\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{int}} = \Omega \hat{a}^\dagger \hat{a} + \sum_q \omega_q \hat{b}_q^\dagger \hat{b}_q \ - i \sum_q [f_q \hat{a}^\dagger \hat{b}_q - f_q^* \hat{a} \hat{b}_q^\dagger] \ . \tag{2}$$

The system Hamiltonian is a simple harmonic oscillator. The bath Hamiltonian is described as a collection of non-interacting harmonic oscillators with frequency ω_q. The interaction Hamiltonian consists of a combination of the annihilation operator and the creation operator of a bath mode, and its complex conjugate. From this starting point, both the Markovian master equation and input output theory can be derived.

1.2 Markovian Master Equation

A widespread approach to describe these open quantum systems is given by Markovian quantum master equations (QME). For all our calculations we will set $\hbar = 1$. The general master equation (which can be found in Ref. [5]) for a system consisting of a coherently driven cavity is given by

$$\partial_t \hat{\rho} = -i[\hat{H}_0 + \hat{H}_d(t), \hat{\rho}] + \kappa n_B D[\hat{a}^\dagger]\hat{\rho} + \kappa(n_B + 1)D[\hat{a}]\hat{\rho} \; . \tag{3}$$

Where $\hat{H}_0 = \Omega \hat{a}^\dagger \hat{a}$ is the free Hamiltonian of the system and

$$\hat{H}_d(t) = i\sqrt{\kappa}\left(\langle \hat{b}_{\text{in}}^\dagger \rangle e^{i\omega_d t}\hat{a} - \langle \hat{b}_{\text{in}} \rangle e^{-i\omega_d t}\hat{a}^\dagger\right) \; , \tag{4}$$

is the drive Hamiltonian. Here $\langle \hat{b}_{\text{in}}^\dagger \rangle$ and $\langle \hat{b}_{\text{in}} \rangle$ are constants and are chosen such that they match later results from input output theory. They can be viewed as the expectation value of the input field in the rotating frame.
Furthermore, $\hat{\rho}$ is the density matrix, κ and n_B are constants that describe the coupling to the heat bath and the Bose-Einstein distribution, respectively and the dissipator operator is given by

$$D[\hat{a}]\hat{\rho} = \hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger \hat{a}, \hat{\rho}\}. \tag{5}$$

To make calculations easier we will transform the QME into a rotating frame, by means of the unitary operator $\hat{U} = e^{i\omega_d \hat{a}^\dagger \hat{a} t}$. This allows us to *rotate* with the drive frequency, making the drive Hamiltonian time-independent. This results in a QME for the rotating frame

$$\begin{aligned} \partial_t \hat{\chi} = &-i\left[\Delta \hat{a}^\dagger \hat{a} + i\sqrt{\kappa}\left(\langle \hat{b}_{\text{in}}^\dagger \rangle \hat{a} - \langle \hat{b}_{\text{in}} \rangle \hat{a}^\dagger\right), \hat{\chi}\right] + \kappa n_B D[\hat{a}^\dagger]\hat{\chi} \\ &+ \kappa(n_B + 1)D[\hat{a}]\hat{\chi} \; , \end{aligned} \tag{6}$$

where $\hat{\chi}$ is the density matrix in the rotating frame and $\Delta = \Omega - \omega_d$ is the detuning. The derivation of this equation can be found in Appendix A

1.3 Quantum Langevin Equations and Input Output Theory

A different approach to describing open quantum systems is through the quantum Langevin equation and input output theory. The derivation of this theory and the Langevin equation along the steps of [6] can be found in Appendix B. The quantum Langevin equation for our system is given by

$$\partial_t \hat{a} = -i\Delta \hat{a} - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\, \hat{b}_{\text{in}}, \tag{7}$$

and yields an equivalent approach to the master equation for the description of our quantum systems. The output of the systems is calculated through the input output relation

$$\hat{b}_{\text{out}} = \hat{b}_{\text{in}} + \sqrt{\kappa}\hat{a} \; , \tag{8}$$

where the input and the output field are defined over the bath variables by Eqs. (106) and (109) in Appendix B. Input output theory offers an elegant way to determine the output of a quantum system, depending entirely on the form of the input driving the system.

1.4 Observables

We can show that the QME and input output theory approaches are equivalent and that from both approaches the same equations of motions for average values can be obtained. From the master equation we can for example obtain the same equation of motion for $\langle\hat{a}\rangle$ as the one that can be derived from the Langevin equation 7, by

$$\partial_t\langle\hat{a}\rangle = \mathrm{Tr}(\hat{a}\dot{\hat{\chi}}) = \left(-i\Delta - \frac{\kappa}{2}\right)\langle\hat{a}\rangle - \sqrt{\kappa}\langle\hat{b}_{\mathrm{in}}\rangle \ . \tag{9}$$

This also justifies our choice to use $\sqrt{\kappa}\langle\hat{b}_{\mathrm{in}}\rangle$ as the constant for the drive Hamiltonian of the master equation. Note that we have used the notation $\dot{\hat{\chi}} = \partial_t\hat{\chi}$ in Eq. 9, for convenience of notation. As our system is Gaussian (the Hamiltonian consist only of terms of first or second order in the creation and annihilation operators), it can be fully described by $\langle\hat{a}\rangle$ and $\langle\hat{a}^\dagger\hat{a}\rangle$. We therefore also derive

$$\partial_t\langle\hat{a}^\dagger\hat{a}\rangle = -\sqrt{\kappa}\left(\langle\hat{b}_{\mathrm{in}}^\dagger\rangle\langle\hat{a}\rangle + \langle\hat{b}_{\mathrm{in}}\rangle\langle\hat{a}^\dagger\rangle\right) - \kappa\langle\hat{a}^\dagger\hat{a}\rangle + \kappa n_B \ . \tag{10}$$

This equation of motion for $\langle\hat{a}^\dagger\hat{a}\rangle$ can be derived from input output theory as well.

1.4.1 Steady State

To find the steady state solution of these differential equations, we simply set them equal to zero, in which case we will have $\partial_t\langle\hat{a}\rangle_{\mathrm{ss}} = 0$ and $\partial_t\langle\hat{a}^\dagger\hat{a}\rangle_{\mathrm{ss}} = 0$. This results in steady state solutions of

$$\langle\hat{a}\rangle_{\mathrm{ss}} = \frac{\sqrt{\kappa}\langle\hat{b}_{\mathrm{in}}\rangle}{(-i\Delta - \frac{\kappa}{2})} = \frac{\sqrt{\kappa}\langle\hat{b}_{\mathrm{in}}\rangle(i\Delta - \frac{\kappa}{2})}{\Delta^2 + (\frac{\kappa}{2})^2} \tag{11}$$

and

$$\langle\hat{a}^\dagger\hat{a}\rangle_{\mathrm{ss}} = n_B + \frac{\kappa|\langle\hat{b}_{\mathrm{in}}\rangle|^2}{\Delta^2 + (\frac{\kappa}{2})^2} \ . \tag{12}$$

Furthermore, we can also have a look at the variance $\langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle_{\mathrm{ss}}$, which is given by

$$\langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle_{\mathrm{ss}} = \langle\hat{a}^\dagger\hat{a}\rangle_{\mathrm{ss}} - \langle\hat{a}^\dagger\rangle_{\mathrm{ss}}\langle\hat{a}\rangle_{\mathrm{ss}} = n_B \tag{13}$$

and only depends on the Bose-Einstein distribution.
As $\langle\hat{a}\rangle_{\mathrm{ss}}$ and $\langle\hat{a}^\dagger\hat{a}\rangle_{\mathrm{ss}}$ completely describe our system in the steady state, we can interpret the results in more detail and see in which domains these results are valid.

We will start with $\langle \hat{a}^\dagger \hat{a} \rangle_{\text{ss}}$, which corresponds to the average of the photon number operator $\hat{N} = \hat{a}^\dagger \hat{a}$. Therefore, $\langle \hat{a}^\dagger \hat{a} \rangle_{\text{ss}}$ describes how many photons are present in our cavity in the steady state. We can see that the term n_B is independent of our drive and therefore is the occupation level of the cavity without a drive, which is expected as it is the Bose-Einstein distribution.

The term $\frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + (\frac{\kappa}{2})^2}$ depends on the frequency and the strength of the input drive as well as the coupling to the heat bath. We see that this term is maximal for $\Delta = 0$, indicating a drive frequency matching the frequency of the cavity. Additionally, we see that the number of photons in the cavity becomes maximal for $\kappa = 0$ as well. This would makes sense, as the dissipation of the system is proportional to κ. However, it is very important to note that for $\kappa = 0$ our equations are no longer valid, as we are not looking at a steady state anymore, because the number of photons in the system would keep growing infinitely. This can be seen in differential Eq. (10), which is not solvable for $\kappa = 0$. Setting $\kappa = 0$, would therefore not correspond to a physical solution.

To illustrate the relation between the steady state photon number $\langle \hat{a}^\dagger \hat{a} \rangle_{\text{ss}}$ and detuning Δ, the photon number N is displayed for different values of $|\langle \hat{b}_{\text{in}} \rangle|^2$ in Fig 2. The figure illustrates that the photon number has a clear maximum at zero detuning and significantly depends on the drive strength. Note that plotting $|\langle \hat{a} \rangle_{\text{ss}}|^2$ would result in the same graph, only shifted down by n_B, as Eq. (13) lets us express $|\langle \hat{a} \rangle_{\text{ss}}|^2$ as

$$|\langle \hat{a} \rangle_{\text{ss}}|^2 = \langle \hat{a}^\dagger \hat{a} \rangle_{\text{ss}} - \langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle_{\text{ss}} = \frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + (\frac{\kappa}{2})^2} \ . \tag{14}$$

To investigate $\langle \hat{a} \rangle_{\text{ss}}$, we look into the scenario of zero detuning $\Delta = 0$, so $\langle \hat{a} \rangle_{\text{ss}}$ can be expressed by a purely reel number. We find

$$\langle \hat{a} \rangle_{\text{ss}} = \frac{-2\langle \hat{b}_{\text{in}} \rangle}{\sqrt{\kappa}} \tag{15}$$

The physical interpretation of $\langle \hat{a} \rangle$ lies in the amplitude of the electric field E and the magnetic field B of the electromagnetic modes inside our cavity and we see that for our system this is linear in the input strength $\langle \hat{b}_{\text{in}} \rangle$ and inversly proporional to $\sqrt{\kappa}$ for the steady state. We are not only interested in the expectation values in the rotating frame, equally important are the expectation values in the lab frame, which is why we have calculated them (see Appendix C) and found expressions of the averages in the lab frame through the averages in the rotating frame

$$\langle \hat{a}^\dagger \hat{a} \rangle_{\text{LAB}} = \langle \hat{a}^\dagger \hat{a} \rangle, \tag{16}$$

and

$$\langle \hat{a} \rangle_{\text{LAB}} = e^{-i\omega_d t} \langle \hat{a} \rangle \ . \tag{17}$$

Note that the average photon number $\langle \hat{a}^\dagger \hat{a} \rangle$ stays invariant under this transformation, whereas the average $\langle \hat{a} \rangle$ attains a time-dependent phase with frequency ω_d, due to the rotation. The expectation values of the output field are of further

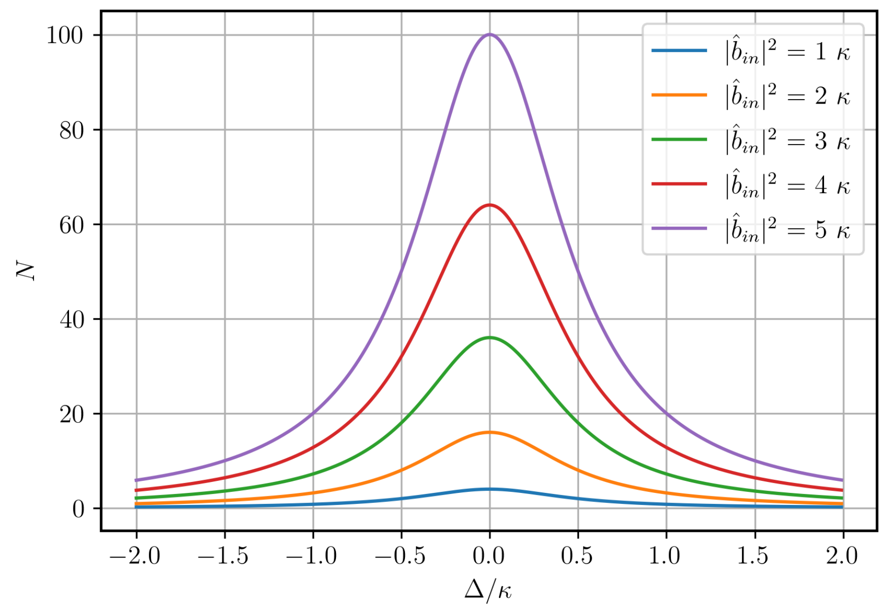


Figure 2: Steady state photon number as a function of detuning $\langle \hat{a}^\dagger \hat{a}\rangle_{\rm ss}(\Delta)$ for different values of the drive strength $|\langle \hat{b}_{\rm in}\rangle|^2$ and constant temperature $n_B = 1$.

interest to us. In Appendix D, we utilize the Langevin equation and transform it to frequency space to solve for our desired quantities. For this calculation it is necessary to specify our input field. In our case this is a coherent drive with thermal noise, which can be represented by

$$\hat{b}_{\rm in}(t) = \langle \hat{b}_{\rm in}\rangle + \hat{\xi}(t)\ , \tag{18}$$

where $\hat{\xi}(t)$ is the thermal noise, with moments

$$\langle \hat{\xi}(t)\rangle = 0 \tag{19}$$

and

$$\langle \hat{\xi}^\dagger(t_1)\hat{\xi}(t_2)\rangle = n_B \delta(t_1 - t_2)\ . \tag{20}$$

The results we get from this approach are

$$\langle \hat{b}_{\rm out}(t)\rangle = \langle \hat{b}_{\rm in}\rangle \frac{i\Delta - \frac{\kappa}{2}}{i\Delta + \frac{\kappa}{2}} \tag{21}$$

and

$$\langle \hat{b}^\dagger_{\rm out}(t)\, \hat{b}_{\rm out}(t')\rangle = |\langle \hat{b}_{\rm in}\rangle|^2 + n_B\delta(t-t') = \langle \hat{b}^\dagger_{\rm in}(t)\, \hat{b}_{\rm in}(t')\rangle\ . \tag{22}$$

We find that the expectation value of the output field is is equal to the expectation value of the input field with a phase shift depending on the detuning Δ.

For a drive frequency that perfectly matches the cavity frequency, we find that the average of the output field is the reflection of the input field, represented by a phase shift of π.
We see that the output photon flux equals the input photon flux, indeed showing that this is a result for steady state operation.

1.4.2 Transient Regime

Now that we have calculated all important quantities for steady state operation, we will proceed to investigate $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \hat{a} \rangle$ in the transient regime. We do so by solving the differential Eqs. (9) and (10). We will solve the equations for an initial vacuum state, which means $\langle \hat{a}(t_0) \rangle = \langle \hat{a}(t_0) \hat{a}(t_0)^\dagger \rangle = 0$. This results in solutions of

$$\langle \hat{a}(t) \rangle = \frac{\sqrt{\kappa} \langle \hat{b}_{\text{in}} \rangle}{-i\Delta - \frac{\kappa}{2}} \left(1 - e^{(-i\Delta - \frac{\kappa}{2})(t - t_0)} \right) \tag{23}$$

and

$$\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = \frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} \left(e^{-\kappa(t - t_0)} - 2e^{-\frac{\kappa}{2}(t - t_0)} \cos(\Delta(t - t_0)) + 1 \right) . \tag{24}$$

The derivation for these results can be found in Appendix E.
We can see that

$$|\langle \hat{a}(t) \rangle|^2 = \langle \hat{a}(t) \rangle \langle \hat{a}^\dagger(t) \rangle = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle, \tag{25}$$

which makes sense, as we set $n_B = 0$ by applying an initial vacuum state. We can plot $\langle \hat{a}^\dagger(t) \hat{a}(t) \rangle$ for multiple values of detuning in the transient regime and normalize this to $A = \frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2}$. This is done in Figure 3.

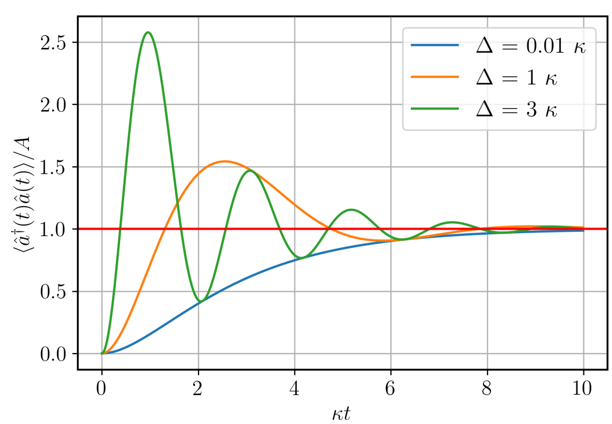


Figure 3: Illustration of $\langle \hat{a}^\dagger \hat{a} \rangle$ in the transient regime for an initial vacuum state with different amounts of detuning $\frac{\Delta}{\kappa}$.

2 Power, Heat and the First Law

The calculation of all the necessary observables allows us to look at the power and heat current for our system. As mentioned before, there is no unambiguous definition for power and heat. We will therefore investigate three different definitions. We will start with the standard definition of power and heat, as it is usually derived from the master equation and will then extend this definition to a modified version, which will be consistent with the master equations as well. Lastly, we will analyze the definition as proposed by [4].

2.1 Standard Definition

In quantum thermodynamics the usual approach to quantify power and heat current follows from the inner energy as demonstrated in [7]

$$U = \mathrm{Tr}(\hat{H}\hat{\rho}) \ , \tag{26}$$

from which we can decompose its partial time derivative as follows:

$$\partial_t U = \underbrace{\mathrm{Tr}(\dot{\hat{H}}\hat{\rho})}_{\mathrm{P}} + \underbrace{\mathrm{Tr}(\hat{H}\dot{\hat{\rho}})}_{\mathrm{J}} = \underbrace{\mathrm{Tr}(\dot{\hat{H}}_d\hat{\rho})}_{\mathrm{P}} + \underbrace{\mathrm{Tr}(\hat{H}_0\dot{\hat{\rho}}) + \mathrm{Tr}(\hat{H}_d\dot{\hat{\rho}})}_{\mathrm{J}} \ . \tag{27}$$

If we substitute our Hamiltonian and density matrix into Eq. (27), we find the power

$$P = -\sqrt{\kappa}\omega_d \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle) + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle \right) \tag{28}$$

and heat current

$$J = \Omega\kappa \left(n_B - \langle \hat{a}^\dagger \hat{a} \rangle \right) + i\frac{\kappa}{2}\sqrt{\kappa} \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle \right) \ . \tag{29}$$

The full derivation can be found in Appendix F. We can see that the heat current consists of all terms that are proportional to κ which come from the dissipator operators and that the power comes from all terms proportional to $\sqrt{\kappa}\langle \hat{b}_{\mathrm{in}} \rangle$, which comes from the drive Hamiltonian. We can now write the power and heat as an expression of the input output operators by using the input output relation

$$\langle \hat{a} \rangle = \frac{\langle \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle}{\sqrt{\kappa}} \ , \tag{30}$$

which results in the power

$$P = -\omega_d \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{b}_{\mathrm{in}} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{b}_{\mathrm{out}}^\dagger \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \right) \tag{31}$$

and heat current

$$\begin{aligned} J = &- \Omega \left(\langle \hat{b}_{\mathrm{out}}^\dagger \, \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{out}}^\dagger \, \hat{b}_{\mathrm{in}} \rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \, \hat{b}_{\mathrm{out}} \rangle + \langle \hat{b}_{\mathrm{in}}^\dagger \, \hat{b}_{\mathrm{in}} \rangle - \kappa n_B \right) \\ &+ i\frac{\kappa}{2} \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{b}_{\mathrm{out}}^\dagger \rangle \right) \ . \end{aligned} \tag{32}$$

2.2 Single Frequency Approach

We are mainly interested in the regime where the drive Hamiltonian only has a small effect compared to the system Hamiltonian and where the detuning is small, meaning $\Omega \approx \omega_d$. We therefore look into a situation where all photons in the system have the energy ω_d. The heat can then be understood as each photon exchanged with the environment contributing with ω_d. This consequently allows us to propose a different definition for the inner energy, namely

$$U = \mathrm{Tr}(\omega_d \hat{a}^\dagger \hat{a} \hat{\rho}) \ . \tag{33}$$

As shown in [8], this approach is thermodynamically consistent if the energy in the Bose-Einstein distribution is given by $\epsilon = \omega_d$. We will also show that the second law of thermodynamics holds for this definition in Sec. 3. This leads to a definition of power

$$P = -\sqrt{\kappa}\omega_d \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle \right) \tag{34}$$

and heat current

$$J = \omega_d \kappa (n_B - \langle \hat{a}^\dagger \hat{a} \rangle) \ . \tag{35}$$

This derivation can be found in Appendix F as well. Note that the definition of power is the exact same as the standard definition. The heat misses a term compared with the standard definition, but this term will be small compared to the rest in our approximation. We can write these definitions in terms of the input output operators as well, giving us

$$P = -\omega_d \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{b}_{\mathrm{in}} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{b}_{\mathrm{out}}^\dagger \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \right) \tag{36}$$

and

$$J = -\omega_d \left(\langle \hat{b}_{\mathrm{out}}^\dagger \hat{b}_{\mathrm{out}} \rangle - \langle \hat{b}_{\mathrm{out}}^\dagger \hat{b}_{\mathrm{in}} \rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \hat{b}_{\mathrm{out}} \rangle + \langle \hat{b}_{\mathrm{in}}^\dagger \hat{b}_{\mathrm{in}} \rangle - \kappa n_B \right) \ . \tag{37}$$

These are the definitions we will be using as the power and heat current derived from the master equations to compare them with the definition by [4].
An overview of both the single frequency approach and the standard definition resulting from the master equation can be found in Table 1 and for steady state power and heat current in Table 2

U	$\mathrm{Tr}(\hat{H}\hat{\rho})$	$\omega_d \mathrm{Tr}(\hat{a}^\dagger \hat{a} \hat{\rho})$
P	$-\sqrt{\kappa}\omega_d(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle) + \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a}^\dagger \rangle)$	$-\sqrt{\kappa}\omega_d(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle) + \langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a}^\dagger \rangle)$
J	$\Omega\kappa(n_B - \langle \hat{a}^\dagger \hat{a} \rangle) + i\frac{\kappa}{2}\sqrt{\kappa}(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle)$	$\omega_d \kappa (n_B - \langle \hat{a}^\dagger \hat{a} \rangle)$

Table 1: Power and heat current for different inner energies.

U	$\mathrm{Tr}(\hat{H}\hat{\rho})$	$\omega_d \mathrm{Tr}(\hat{a}^\dagger \hat{a}\hat{\rho})$
P_{ss}	$\omega_d \frac{\kappa^2 \lvert\langle \hat{b}_{\mathrm{in}}\rangle\rvert^2}{\Delta^2+(\frac{\kappa}{2})^2}$	$\omega_d \frac{\kappa^2 \lvert\langle \hat{b}_{\mathrm{in}}\rangle\rvert^2}{\Delta^2+(\frac{\kappa}{2})^2}$
J_{ss}	$-\omega_d \frac{\kappa^2 \lvert\langle \hat{b}_{\mathrm{in}}\rangle\rvert^2}{\Delta^2+(\frac{\kappa}{2})^2}$	$-\omega_d \frac{\kappa^2 \lvert\langle \hat{b}_{\mathrm{in}}\rangle\rvert^2}{\Delta^2+(\frac{\kappa}{2})^2}$

Table 2: Power and heat current for different inner energies.

2.3 More Energy Flows and the First Law

As we are looking into power and heat, which are different energy flows, it is also interesting to see how the different energies of our Hamiltonians change. We will therefore take a closer look at the equation of motion for $\langle \hat{H}_{\mathrm{bath}}\rangle$, as defined in Eq. (2):

$$\partial_t \langle \hat{H}_{\mathrm{bath}}\rangle = \left\langle i[\hat{H}, \hat{H}_{\mathrm{bath}}]\right\rangle = \left\langle \sum_q \omega_q \left(f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a}\hat{b}_q^\dagger \right)\right\rangle \ . \tag{38}$$

We can do the same for the other two Hamiltonians and find

$$\partial_t \langle \hat{H}_{\mathrm{sys}}\rangle = -\left\langle \Omega \sum_q [f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a}\hat{b}_q^\dagger] \right\rangle \tag{39}$$

and

$$\partial_t \langle \hat{H}_{\mathrm{int}}\rangle = \left\langle \Delta \sum_q [f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a}\hat{b}_q^\dagger] \right\rangle \ , \tag{40}$$

resulting in $\partial_t \langle \hat{H}\rangle = \partial_t \langle \hat{H}_{\mathrm{sys}} + \hat{H}_{\mathrm{bath}} + \hat{H}_{\mathrm{int}}\rangle = 0$, which is expected because of energy conservation. We can make the Markovian approximation again and approximate these expressions in terms of input output operators as well to find

$$\partial_t \langle \hat{H}_{\mathrm{bath}}\rangle = \omega_d \left(\langle \hat{b}_{\mathrm{out}}^\dagger \, \hat{b}_{\mathrm{out}}\rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \, \hat{b}_{\mathrm{in}}\rangle \right) \ , \tag{41}$$

$$\partial_t \langle \hat{H}_{\mathrm{sys}}\rangle = -\Omega \left(\langle \hat{b}_{\mathrm{out}}^\dagger \, \hat{b}_{\mathrm{out}}\rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \, \hat{b}_{\mathrm{in}}\rangle \right) \ , \tag{42}$$

$$\partial_t \langle \hat{H}_{\mathrm{bath}}\rangle = \Delta \left(\langle \hat{b}_{\mathrm{out}}^\dagger \, \hat{b}_{\mathrm{out}}\rangle - \langle \hat{b}_{\mathrm{in}}^\dagger \, \hat{b}_{\mathrm{in}}\rangle \right) \ . \tag{43}$$

The derivation of these energy flows and the calculation for input output theory can be found in Appendix G.

We also want to look at the first law of thermodynamics, which says that the total amount of energy is conserved or that energy cannot be created nor destroyed. We can formulate this by adding the different energy flows to equal zero. For any definition of power and heat, we should therefore find that they obey

$$P + J = -\partial_t \langle \hat{H}_{\mathrm{bath}}\rangle \ , \tag{44}$$

within the approximations of $\omega_d \approx \Omega$.

2.4 Definition based on Input and Output

We will now look into the definition made by Ref. [4], where work is defined as the coherent part of the output. As shown in this paper, a quantum battery prepared in a coherent state is able to perform thermodynamic work. Hence, work is defined as the coherent part of the output field and heat is defined as the incoherent part of the output, which leads to the mathematical definition of work as

$$P' = -\omega_d \left(\langle \hat{b}^\dagger_{\text{out}} \rangle \langle \hat{b}_{\text{out}} \rangle - \langle \hat{b}^\dagger_{\text{in}} \rangle \langle \hat{b}_{\text{in}} \rangle \right) \ . \tag{45}$$

Heat current is consequently defined over energy conservation and may be expressed as

$$J' = -\left(\partial_t \langle \hat{H}_{\text{bath}} \rangle + P' \right) \ . \tag{46}$$

This results in a heat current of

$$J' = -\omega_d \left(\langle\langle \hat{b}^\dagger_{\text{out}} \, \hat{b}_{\text{out}} \rangle\rangle - \langle\langle \hat{b}^\dagger_{\text{in}} \, \hat{b}_{\text{in}} \rangle\rangle \right) \ , \tag{47}$$

where $\langle\langle \hat{A}\hat{B} \rangle\rangle = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$ is the covariance of $\hat{A}\hat{B}$.

2.5 Comparison

We are interested in the differences between the energy flows defined in terms of the framework used for the QME and the energy flows defined over the in- and output. Therefore, we can equate both definitions of power through

$$P' = P - \omega_d \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \tag{48}$$

by using the input output relations. We can also compare both definitions of heat current by using the relation

$$\langle\langle \hat{b}^\dagger_{\text{out}} \, \hat{b}_{\text{out}} \rangle\rangle = \langle\langle \hat{b}^\dagger_{\text{in}} \, \hat{b}_{\text{in}} \rangle\rangle + \kappa \left[\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle - n_B \right] \ , \tag{49}$$

which is derived in Appendix H. From these relations, we can express J' as

$$J' = -\omega_d \kappa \left[\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle - n_B \right] \ , \tag{50}$$

which allows us to compare J and J' through

$$J' = J + \omega_d \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \ . \tag{51}$$

As J' is defined over energy conservation and $P + J = P' + J'$, we see that the first law (44) is automatically verified for P and J as well.

2.6 Energy Flows in the Transient Regime

To get a better view of the dynamics of the system, we will investigate the power, heat and bath energy in the transient regime, starting from the vacuum state. For this we will use the solutions of $\langle \hat{a}(t) \rangle$ and$\langle \hat{a}^\dagger \hat{a}(t) \rangle$ for the transient regime, as calculated in Sec. 1.4.2. We will start with the calculation for the power.

2.6.1 Single Frequency Definition of Power P

We can use the definition of power in Eq. (34) and substitute our solutions for $\langle \hat{a}(t) \rangle$ and $\langle \hat{a}(t)^\dagger \rangle$, to find that we can express the power for an initial vacuum state as

$$P = \omega_d \frac{\kappa^2 |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} \left(1 - e^{-\frac{\kappa}{2}(t-t_0)} (\cos(\Delta(t-t_0)) - 2\frac{\Delta}{\kappa} \sin(\Delta(t-t_0))) \right), \quad (52)$$

which can be expressed in terms of the steady state power as

$$P = P_{ss} \left(1 - e^{-\frac{\kappa}{2}(t-t_0)} (\cos(\Delta(t-t_0)) - 2\frac{\Delta}{\kappa} \sin(\Delta(t-t_0))) \right). \quad (53)$$

When driven on the resonance frequency ($\Delta = 0$), this results in:

$$P_{\Delta=0} = 4\omega_d |\langle \hat{b}_{\text{in}} \rangle|^2 \left(1 - e^{-\frac{\kappa}{2}(t-t_0)} \right) \quad (54)$$

and when we remove the drive we find the trivial relation $P = 0$. We can plot P to see how the Power changes before settling to the steady state. This is illustrated in Fig. 4a for different amounts of detuning Δ.

2.6.2 Definition of Power based on Output P'

We can also look at the definition of power based on the output P', from [4], which can be written as:

$$\begin{aligned} P' &= -\omega_d (\langle \hat{b}^\dagger_{\text{out}} \rangle \langle \hat{b}_{\text{out}} \rangle - \langle \hat{b}^\dagger_{\text{in}} \rangle \langle \hat{b}_{\text{in}} \rangle) = -\omega_d (\sqrt{\kappa} (\langle \hat{b}^\dagger_{\text{in}} \rangle \langle \hat{a} \rangle + \langle \hat{b}_{\text{in}} \rangle \langle \hat{a}^\dagger \rangle) + \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle) \\ &= P - \omega_d \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \end{aligned} \quad (55)$$

By substituting in $\langle \hat{a} \rangle$ and $\langle \hat{a}^\dagger \rangle$, we can express P' through

$$\begin{aligned} P' =& \omega_d \frac{\kappa^2 |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} \Big(e^{-\frac{\kappa}{2}(t-t_0)} (\cos(\Delta(t-t_0)) \\ &+ 2\frac{\Delta}{\kappa} \sin(\Delta(t-t_0))) - e^{-\kappa(t-t_0)} \Big). \end{aligned} \quad (56)$$

We can also normalize this by the steady state power of the single frequency definition. This results in

$$P' = P_{ss}\bigg(e^{-\frac{\kappa}{2}(t-t_0)}(\cos(\Delta(t-t_0)) + 2\frac{\Delta}{\kappa}\sin(\Delta(t-t_0))) - e^{-\kappa(t-t_0)}\bigg). \quad (57)$$

Here we can also look at the resonance case ($\Delta = 0$), resulting in

$$P'_{\Delta=0} = 4\omega_d|\langle\hat{b}_{\text{in}}\rangle|^2\bigg(e^{-\frac{\kappa}{2}(t-t_0)} - e^{-\kappa(t-t_0)}\bigg). \quad (58)$$

This power in the transient regime is plotted in Figure 4b. We can see that the definition of the power by the single frequency approach tends to a non-zero steady state after some initial oscillations. The power based on the definition with input output theory, however, tends to a steady state of zero after some similar initial oscillations.

2.6.3 Single Frequency Definition of Heat Current J

We can do the same for the heat current $J = \omega_d\kappa(n_B - \langle\hat{a}^\dagger\hat{a}\rangle)$. With our solution for $\langle\hat{a}^\dagger\hat{a}\rangle$, we get:

$$J = J_{ss}\big(e^{-\kappa(t-t_0)} - 2e^{-\frac{\kappa}{2}(t-t_0)}\cos(\Delta(t-t_0))\big) + 1\big) \quad (59)$$

and when driven on the resonance frequency ($\Delta = 0$), we get:

$$J_{\Delta=0} = -4\omega_d|\langle\hat{b}_{\text{in}}\rangle|^2\big(e^{-\kappa(t-t_0)} - 2e^{-\frac{\kappa}{2}(t-t_0)} + 1\big). \quad (60)$$

We see that without a drive ($|\langle\hat{b}_{\text{in}}\rangle|^2 = 0$), the Heat current disappears for an intial vacuum state

$$J_{\hat{b}_{\text{in}}=0} = 0. \quad (61)$$

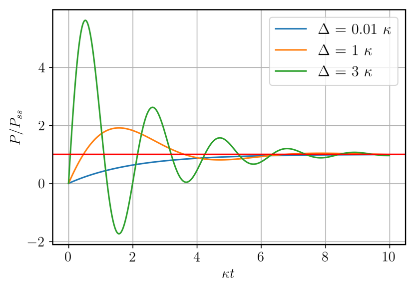
We can also plot J for different amounts of detuning Δ, which can be seen in Figure 4c.

2.6.4 Definition of Heat Current based on Output J'

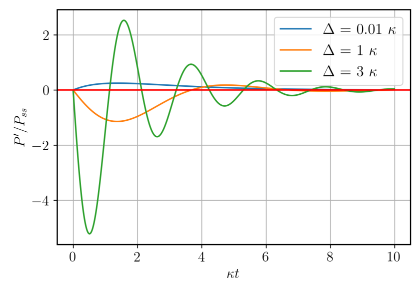
We can use the comparison between the two definitions of heat current to find

$$J' = J + \omega_d\kappa\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle = 0. \quad (62)$$

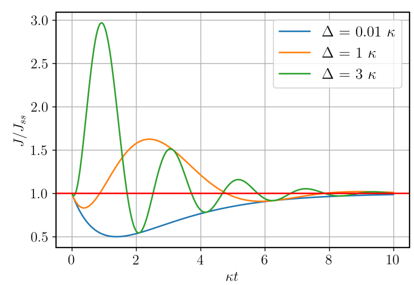
We find that this heat current vanishes for zero temperature ($n_B = 0$), when we start in the vacuum state.



(a) Power(single frequency appr.)



(b) Power (input output theory)



(c) Heat current (single frequency appr.)

Figure 4: Definitions of power and heat in the transient regime for an initial vacuum state with different amounts of detuning $\frac{\Delta}{\kappa}$. The power and heat currents tends to their respective steady state values.

2.6.5 Plotting the Energy Flows for Driving at Resonance Frequency

Lastly, we are interested to see how the four plots for the energy flows look like, when the cavity is driven at its resonance frequency ($\Delta = 0$). This is can be seen in Figure 5. The expressions for the energy flows can be found below

$$P_{\Delta=0} = 4\omega_d |\langle \hat{b}_{\text{in}} \rangle|^2 \left(1 - e^{-\frac{\kappa}{2}(t-t_0)}\right), \tag{63}$$

$$J_{\Delta=0} = -4\omega_d |\langle \hat{b}_{\text{in}} \rangle|^2 \left(e^{-\kappa(t-t_0)} - 2e^{-\frac{\kappa}{2}(t-t_0)} + 1\right), \tag{64}$$

$$P'_{\Delta=0} = 4\omega_d |\langle \hat{b}_{\text{in}} \rangle|^2 \left(e^{-\frac{\kappa}{2}(t-t_0)} - e^{-\kappa(t-t_0)}\right), \tag{65}$$

$$J'_{\Delta=0} = 0. \tag{66}$$

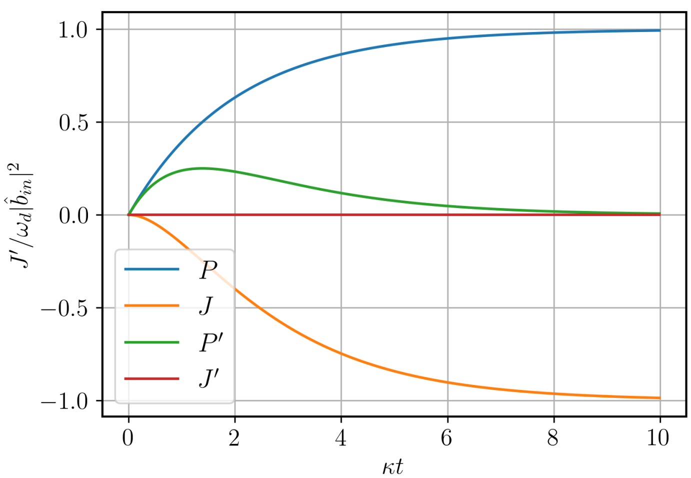


Figure 5: Energy flows for the single frequency definitions (P and J) and the definition based on the output (P' and J') in the transient regime for $\Delta = 0$ with an initial vacuum state.

3 Entropy and the Second Law

To further investigate the differences between the definitions of power and heat current through the master equations and the definition for input output theory, we will look at the entropy production rate. This allows us to see whether the definitions made by [4] are suited to form a consistent set of thermodynamic variables including a consistent set of thermodynamic laws to describe systems where the output is not completely discarded. As we have already shown that the first law is verified for the definitions of power and heat and the definition by [4] in Sec. 2.5, we will now look into the second law. In particular we will investigate whether the second law of thermodynamics, which says that entropy can never decrease, holds for the single frequency definitions of power and heat and the definition by [4].

3.1 Entropy Production

We introduce the entropy production rate

$$\partial_t \Sigma = \frac{-J}{T} + k_B \partial_t S_{\mathrm{vN}}[\hat{\rho}] \ , \tag{67}$$

where J is our heat current, T is the temperature, k_B is the Boltzmann constant, S_{vN} the von Neumann entropy and $\hat{\rho}$ the density matrix of the system. In Appendix J, we calculate the entropy production rate, according to [9], for a Gaussian state with no squeezing, for our heat current J. We express T through the Bose-Einstein distribution where we have used $\epsilon = \omega_d$. We find

$$\begin{aligned} \partial_t \Sigma =& k_B \kappa \left(n_B - \langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle \right) \left(\ln\left(\frac{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle + 1}{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle} \right) - \ln\left(\frac{n_B + 1}{n_B} \right) \right) \\ &+ k_B \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \ln\left(\frac{n_B + 1}{n_B} \right) \ . \end{aligned} \tag{68}$$

By closer investigation of this result we can see that the first term is bigger than zero

$$k_B \kappa \left(n_B - \langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle \right) \left(\ln\left(\frac{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle + 1}{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle} \right) - \ln\left(\frac{n_B + 1}{n_B} \right) \right) \geq 0 \ , \tag{69}$$

because when n_B is bigger than $\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle$, we find

$$\ln\left(\frac{n_B + 1}{n_B} \right) \leq \ln\left(\frac{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle + 1}{\langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle} \right) , \tag{70}$$

as $\ln\left(\frac{x+1}{x}\right)$ is a monotonically decreasing function for increasing x, and vice versa. We also see that the second term is bigger than zero

$$k_B \kappa \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \ln\left(\frac{n_B + 1}{n_B} \right) = k_B \kappa |\langle \hat{a} \rangle|^2 \ln\left(\frac{n_B + 1}{n_B} \right) \geq 0 \ . \tag{71}$$

This ultimately results in an entropy production that is greater or equal to zero

$$\partial_t \Sigma \geq 0 \tag{72}$$

and verifies the second law for our system with the single frequency definition of power and heat current.

3.2 Entropy Production for J'

Now that we have calculated the entropy production rate for J and have shown that the second law of thermodynamics holds for this example, we will do the same for J'. This results in an entropy production rate of

$$\partial_t \Sigma' = k_B \kappa \left(n_B - \langle\langle \hat{a}^\dagger \hat{a} \rangle\rangle\right) \left(\ln\left(\frac{\langle\langle \hat{a}^\dagger \hat{a}\rangle\rangle + 1}{\langle\langle \hat{a}^\dagger \hat{a}\rangle\rangle} \right) - \ln\left(\frac{n_B + 1}{n_B} \right) \right) , \tag{73}$$

which is also calculated in Appendix J. Similar to the entropy production for J, we find that the second law ($\partial_t \Sigma' \geq 0$) also holds for this definition of J'.
Contrary to the definition of J in the master equation approach, not everything that leaves the system is regarded as dissipation for the definition J' of input output theory, but only the incoherent part of the output is regarded as dissipation. This results in a smaller entropy production because there is less dissipation.

3.3 Entropy Production of the Bath

Now that we have shown that the definition of power and heat current made by [4] are suitable to be part of a set of thermodynamic quantities with a first and second law for a simple harmonic oscillator, we want to understand how general these result are. Therefore we will investigate the entropy production of the bath to see if we can prove that the first and second law will hold for any general system. We will do this by looking at the total change in entropy

$$\partial_t S_{\mathrm{tot}} \equiv \partial_t S_{\mathrm{vN}}[\rho_B] + \partial_t S_{\mathrm{vN}}[\rho] \geq 0 , \tag{74}$$

which will always be greater or equal to zero. This is because the von Neumann entropy of the entire system stays constant

$$\partial_t S_{\mathrm{vN}}[\rho_{\mathrm{tot}}] = \partial_t S_{\mathrm{vN}}[\rho_B] + \partial_t S_{\mathrm{vN}}[\rho] - \partial_t I_{\mathrm{MI}}[S; B] = 0 , \tag{75}$$

because of unitary time evolution and the mutual information of the bath and the system $I_{\mathrm{MI}}[S;B]$ increase over time. Here, ρ_B is the density matrix of the bath and the mutual information is defined by

$$I_{\mathrm{MI}}[S; B] = S_{\mathrm{vN}}[\rho_{\mathrm{S}}] + S_{\mathrm{vN}}[\rho_{\mathrm{B}}] - S_{\mathrm{vN}}[\rho_{\mathrm{SB}}], \tag{76}$$

Where $\rho_S = \rho$ is the density matrix of the system and ρ_{SB} is the density matrix of the systen combined with the bath.
We can therefore show that entropy production rate

$$\partial_t \Sigma' = \frac{-J'}{T} + k_B \partial_t S_{\mathrm{vN}}[\hat{\rho}] \geq 0 , \tag{77}$$

is greater than or equal to zero, whenever

$$\frac{-J'}{T} \geq k_B \partial_t S_{\mathrm{vN}}[\rho_B]. \tag{78}$$

Note that J' is defined over energy conservation and the definition of the power and therefore is independent of the system.

4 Conclusion and Outlook

In this thesis we have looked at how power and heat can be defined, motivated by two theories regarding open quantum systems; the Markovian master equation and input output theory. We investigated and quantified the differences between both theories for a simple harmonic oscillator. We started by the widespread approach of the master equation and determined power and heat for our system. From there, we wrote these quantities in terms of the central objects of input output theory, which resulted in a definition of heat and work for input output theory consistent with the master equations.
A different definition of heat and work for input output theory given by [4] sparked motivation to investigate the discrepancy of these two definitions further. We have shown that both the approach of the master equation and the approach by [4], considering the coherent part of the output as work, can be used to consistently describe the thermodynamics of the system. Furthermore, we have shown that both approaches can be used in combination with a consistent first and second law of thermodynamics, which we have defined over the entropy production rate.
The definitions of this new approach by [4] and the associated thermodynamic laws we found are therefore candidates in the construction of a new thermodynamic framework describing the output of open quantum systems. Such a framework would be used whenever the output of the system is of interest and can be controlled. It would view part of the output as work, which is fundamentally different to the current approach of the Markovian master equation, where everything leaving the system is regarded as lost.
A concrete example where this new framework could yield new and different results is in the TUR (Thermodynamic Uncertainty Relation) violation of 3 level thermodynamic masers. Here the dimensionless quantity $Q = \frac{\sigma}{k_b} \frac{\langle\langle P \rangle\rangle}{\langle P \rangle^2} \geq 2$ is bound from below by two. As shown in Ref. [10], the TUR is violated for 3 level thermodynamic lasers, when the system is treated semi-classically, but the violation is no longer there when treated quantum mechanically. With our new approach this violation may however be restored, when the system is treated quantum mechanically.
It is important to note that we have only shown that these definitions and thermodynamic laws build a consistent framework for a simple harmonic oscillator model and not for a general case. Showing that his is the case for a general system, would be the next step towards a completely general framework to describe the thermodynamics of the output of open quantum system.

A Master Equation in the Rotating Frame

To solve the Master equation (3), reproduced here for convenience

$$\partial_t \hat{\rho} = -i\left[\hat{H}_0 + \hat{H}_d(t), \hat{\rho}\right] + \kappa n_B D[\hat{a}^\dagger]\hat{\rho} + \kappa(n_B+1)D[\hat{a}]\hat{\rho} \ , \tag{79}$$

it is convenient to transform $\hat{\rho}$ into a so called rotating frame by means of the unitary operator $\hat{U} = e^{i\omega_d \hat{a}^\dagger \hat{a} t}$. This allows us to rotate with the drive frequency, making the drive Hamiltonian time-independent. We can calculate the density matrix in this rotating frame, which we define by $\hat{\chi} = \hat{U}^\dagger \hat{\rho} \hat{U}$. From this follows

$$\dot{\hat{\chi}} = \dot{\hat{U}}^\dagger \hat{\rho} \hat{U} + \hat{U}^\dagger \dot{\hat{\rho}} \hat{U} + \hat{U}^\dagger \hat{\rho} \dot{\hat{U}} \ . \tag{80}$$

With a short calculation, we can express $\dot{\hat{\chi}}$ through

$$\dot{\hat{\chi}} = \dot{\hat{U}}^\dagger \hat{U} \hat{\chi} + \hat{\chi} \hat{U}^\dagger \dot{\hat{U}} + \hat{U}^\dagger \dot{\hat{\rho}} \hat{U} = \left[\dot{\hat{U}}^\dagger \hat{U}, \hat{\chi}\right] + \hat{U}^\dagger \dot{\hat{\rho}} \hat{U} \ . \tag{81}$$

If we now substitute our Master Eq. (79) into our expression for $\dot{\hat{\chi}}$ in Eq. (81), we get a Master equation in the rotating frame

$$\dot{\hat{\chi}} = -i[\tilde{H}, \hat{\chi}] + \kappa n_B D[\hat{U}^\dagger \hat{a}^\dagger \hat{U}]\hat{\chi} + \kappa(n_B+1)D[\hat{U}^\dagger \hat{a} \hat{U}]\hat{\chi} \ , \tag{82}$$

where $\tilde{H} = \hat{U}^\dagger \hat{H}_0 \hat{U} + \hat{U}^\dagger \hat{H}_d(t) \hat{U} + i\dot{\hat{U}}^\dagger \hat{U}$ takes the place of the Hamiltonian. We can now evaluate the terms of this Hamiltonian explicitly, which starting with the first term yields the following result

$$\hat{U}^\dagger \hat{H}_0 \hat{U} = \Omega e^{i\omega_d \hat{a}^\dagger \hat{a} t} \hat{a}^\dagger \hat{a} e^{-i\omega_d \hat{a}^\dagger \hat{a} t} = \Omega \hat{a}^\dagger \hat{a} \ . \tag{83}$$

Here we have used the identity $[e^{\hat{O}}, \hat{O}] = 0$ to transform $\hat{H}_0$ into the rotating frame. Note that $\hat{a}^\dagger \hat{a}$ stays invariant under this transformation, as it is an expression of the amount of photos in our system. For the drive Hamiltonian in the rotating frame we find the expression

$$\hat{U}^\dagger \hat{H}_d(t) \hat{U} = i\sqrt{\kappa}\left(\langle \hat{b}_{\text{in}}^\dagger \rangle e^{i\omega_d t} \hat{U}^\dagger \hat{a} \hat{U} - \langle \hat{b}_{\text{in}} \rangle e^{-i\omega_d t} \hat{U}^\dagger \hat{a}^\dagger \hat{U}\right) \ , \tag{84}$$

where we can evaluate $\hat{U}^\dagger \hat{a} \hat{U}$ and $\hat{U}^\dagger \hat{a}^\dagger \hat{U}$ to be $e^{-i\omega_d t}\hat{a}$ and $e^{i\omega_d t}\hat{a}^\dagger$ respectively. This is done with the Hadamard lemma, resulting from the Baker-Campbell-Hausdorff-Formula

$$e^X Y e^{-X} = \sum_{m=0}^{\infty} \frac{[X,Y]_m}{m!} \ , \tag{85}$$

with $[X,Y]_m = [X,[X,Y]_{m-1}]$ and $[X,Y]_0 = Y$. With these identities we can write the Master equation in the rotating frame (Eq. (82)) in an explicit form for our Hamiltonians and we obtain

$$\begin{aligned} \dot{\hat{\chi}} =& -i\left[\Omega \hat{a}^\dagger \hat{a} + i\sqrt{\kappa}\left(\langle \hat{b}_{\text{in}}^\dagger \rangle \hat{a} - \langle \hat{b}_{\text{in}} \rangle \hat{a}^\dagger\right) - \omega_d \hat{a}^\dagger \hat{a}, \hat{\chi}\right] \\ &+ \kappa n_B D[\hat{a}^\dagger]\hat{\chi} + \kappa(n_B+1)D[\hat{a}]\hat{\chi} \ . \end{aligned} \tag{86}$$

This can be simplified by introducing the detuning $\Delta := \Omega - \omega_d$, describing the difference between the resonance frequency of the cavity and the drive frequency, resulting in

$$\begin{aligned}\dot{\hat{\chi}} =& -i\left[\Delta\hat{a}^\dagger\hat{a} + i\sqrt{\kappa}\left(\langle\hat{b}_{\text{in}}^\dagger\rangle\hat{a} - \langle\hat{b}_{\text{in}}\rangle\hat{a}^\dagger\right), \hat{\chi}\right] + \kappa n_B D[\hat{a}^\dagger]\hat{\chi} \\ &+ \kappa(n_B+1)D[\hat{a}]\hat{\chi} \ ,\end{aligned} \tag{87}$$

which is the final form we will be using for our calculations.

B Derivation of Input Output Theory

In this section we will show the derivation of the theory known as input output theory. In this theory we calculate the output of our system through the interaction of the system with the input it is given. We will derive this theory following the same steps as [6]. We are dealing with an open system in a cavity, which is coupled to a heat bath and is driven by an external source. Therefore we can write the following Hamiltonian from Eq. (2), reproduced here for convenience:

$$\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{int}} \ . \tag{88}$$

In which the system's Hamiltonian is not yet specified (but will only be proportional to $\hat{a}$ and $\hat{a}^\dagger$) and the bath- and interaction Hamiltonians:

$$\hat{H}_{\text{bath}} = \sum_q \omega_q \hat{b}_q^\dagger \hat{b}_q \quad , \tag{89}$$

$$\hat{H}_{\text{int}} = -i\sum_q \left[f_q\hat{a}^\dagger\hat{b}_q - f_q^*\hat{a}\hat{b}_q^\dagger\right] \quad . \tag{90}$$

Contrary to the calculations for the master equation, we will now be working in the Heisenberg picture instead of the Schrödinger picture. Therefore our operators will now become time-dependent. However, for convenience of notation and readability, we will often drop the time argument of the operators. From these Hamiltonians we can write down the Heisenberg equation of motion for our bath variables

$$\partial_t\hat{b}_q = [\hat{H}, \hat{b}_q] \ , \tag{91}$$

which, by using the identities $[\hat{b}_q, \hat{b}_q'^\dagger] = \delta_{q,q'}$ and $[\hat{b}_q, \hat{a}] = [\hat{b}_q, \hat{a}^\dagger] = [\hat{b}_q, \hat{b}_q'] = 0$, we can evaluate to be

$$\dot{\hat{b}}_q = i\left[\omega\hat{b}_q^\dagger\hat{b}_q + if_q^*\hat{a}_q\hat{b}_q^\dagger, \hat{b}_q\right] = -i\omega_q\hat{b}_q + f_q^*\hat{a} \ . \tag{92}$$

This corresponds to a differential equation of a driven harmonic oscillator, which can be solved by

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) + \int_{t_0}^t e^{-i\omega_q(t-\tau)} f_q^*\hat{a}(\tau)\, d\tau \ . \tag{93}$$

The first term of this expression represents the free evolution of $\hat{b}_q(t)$, whereas the second term expresses the waves radiated from the cavity into the bath.
If we now do the same for $\hat{a}$, we get find that its equation of motion is given by

$$\partial_t \hat{a} = i[\hat{H}, \hat{a}] = i[\hat{H}_{\text{sys}}, \hat{a}] + \sum_q \left[f_q \hat{a}^\dagger \hat{b}_q - f_q^* \hat{a} \hat{b}_q^\dagger, \hat{a} \right] \quad . \tag{94}$$

By using the identities $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{b}_q, \hat{a}] = [\hat{b}_q^\dagger, \hat{a}] = 0$, we can simplify this again, resulting in

$$\partial_t \hat{a} = i[\hat{H}_{\text{sys}}, \hat{a}] - \sum_q f_q \; \hat{b}_q \; . \tag{95}$$

To further evaluate this result, we can use Eq. (93) to express $\sum_q f_q \hat{b}_q$ explicitly. We find the expression

$$\sum_q f_q \hat{b}_q = \sum_q f_q e^{-i\omega_q (t-t_0)} \hat{b}_q(t_0) \; + \sum_q |f_q|^2 \int_{t_0}^{t} e^{-i\omega_q (t-\tau)} \hat{a}(\tau) \, d\tau \quad . \tag{96}$$

This can be written as

$$\begin{aligned} \sum_q f_q \hat{b}_q = & \sum_q f_q e^{-i\omega_q (t-t_0)} \hat{b}_q(t_0) \\ & + \sum_q |f_q|^2 \int_{t_0}^{t} e^{-i(\omega_q - \omega_c)(t-\tau)} [e^{i\omega_c(\tau - t)} \hat{a}(\tau)] \, d\tau \quad , \end{aligned} \tag{97}$$

in which we assume the term $[e^{i\omega_c(\tau-t)}\hat{a}(\tau)]$ to be a slowly varying function of τ, as $\hat{a}$ would evolve with frequency ω_c if it were a single harmonic oscillator. This allows for the Markov approximation, allowing only bath modes with frequencies close to ω_c to interact with the bath.
We can now define κ as

$$\kappa(\omega_c) = 2\pi \sum_q |f_q|^2 \delta(\omega_c - \omega_q) \quad . \tag{98}$$

From this definition follows the relation:

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \kappa(\omega_c + \nu) e^{-i\nu(t-\tau)} \, d\nu \; = \sum_q |f_q|^2 \; e^{-i(\omega_q - \omega_c)(t-\tau)} \; . \tag{99}$$

We can now make the Markov approximation $\kappa(\nu) = \kappa$, in which we assume that κ is independent of the frequency, so that we can express Eq. (99) as

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \kappa(\omega_c + \nu) e^{-i\nu(t-\tau)} \, d\nu \; = \kappa \delta(t - \tau) \; . \tag{100}$$

We can now set the right sides of Eq, (99) and (100) equal, which gives us the relation

$$\sum_q |f_q|^2 \; e^{-i(\omega_q - \omega_c)(t-\tau)} = \kappa \delta(t - \tau) \; . \tag{101}$$

We can substitute this expression back in Eq. (97), to find an expression for $\sum_q f_q \hat{b}_q$:

$$\sum_q f_q \hat{b}_q = \sum_q f_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) \ + \sum_q \int_{t_0}^{t} \kappa\delta(t-\tau)[e^{i\omega_c(\tau-t)}\hat{a}](\tau)\, d\tau \ \ . \tag{102}$$

Here we can evaluate $\int_{t_0}^{t} \kappa\delta(t-\tau)[e^{i\omega_c(\tau-t)}\hat{a}](\tau)\, d\tau$ to be $\frac{\kappa}{2}\hat{a}$, using the identity

$$\int_{t_0}^{t} \delta(t-\tau)\, d\tau \ = \frac{1}{2} \ , \tag{103}$$

resulting in the differential equation for a:

$$\partial_t \hat{a} = i[\hat{H}_{\text{sys}}, \hat{a}] - \frac{\kappa}{2}\hat{a} - \sum_q f_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) \ \ , \tag{104}$$

where, the $\frac{\kappa}{2}\hat{a}$ term represents the damping of the cavity. A further approximation to simplify calculations can be made by defining f as $f := \sqrt{|f_q|^2}$ and the density of states as $\rho = \sum_q \delta(\omega_c - \omega_q)$. This leads to κ being expressed through

$$\kappa = 2\pi|f|^2\rho \ . \tag{105}$$

By making these approximations, we can now evaluate the equation of motion for $\hat{a}$ further. In particular, we can define the 'input mode' $\hat{b}_{\text{in}}$, from the third term in Eq. (104) as

$$\hat{b}_{\text{in}} = \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) \ \ . \tag{106}$$

This input field can be interpreted as a field moving towards the cavity. This field is evolving freely under the bath Hamiltonian and is the driving term of our cavity. With this we can rewrite Eq. (104) as:

$$\partial_t \hat{a} = i[\hat{H}_{\text{sys}}, \hat{a}] - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\, \hat{b}_{\text{in}}(t) \ . \tag{107}$$

We now have an input mode coming from the bath interacting with the cavity. Additionally, we can obtain a differential equation for the cavity modes through this input mode and the system Hamiltonian. This interaction yields an outgoing field, which we will describe as $\hat{b}_{\text{out}}$. Similar to $\hat{b}_{\text{in}}$, this field will emerge in the solution of Eq. (92) as well, with the only difference that this solution to the equation is now described by the final conditions, rather than the initial:

$$\hat{b}_q(t) = e^{-i\omega_q(t-t_1)}\hat{b}_q(t_0) - \int_{t}^{t_1} e^{-i\omega_q(t-\tau)} f_q^* \hat{a}(\tau)\, d\tau \ . \tag{108}$$

By following the same steps as above, this gives us the definition of $\hat{b}_{\text{out}}$ as

$$\hat{b}_{\text{out}} = \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{-i\omega_q(t-t_1)}\hat{b}_q(t_1) \ \ . \tag{109}$$

Similar to $\hat{b}_{\text{in}}$, we can also write Eq. (104) in terms of $\hat{b}_{\text{out}}$, instead of $\hat{b}_{\text{in}}$, which would result in

$$\partial_t \hat{a} = i[\hat{H}_{\text{sys}}, \hat{a}] + \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\,\hat{b}_{\text{out}} \ . \tag{110}$$

Comparing Eq. (110) with Eq. (107), gives us the interesting insight that

$$\boxed{\hat{b}_{\text{out}} = \hat{b}_{\text{in}} + \sqrt{\kappa}\hat{a}} \ . \tag{111}$$

This is a key insight to input output theory, as it lets us determine the outgoing field, depending only on the incoming field and the operator $\hat{a}$, which can be completely determined by the system Hamiltonian and again the incoming field.

C Expectation Values in the Lab Frame

As we are mainly interested in our quantities in the LAB frame, but solve our problem in the rotating frame, we provide the expectation values for $\hat{a}$ and $\hat{a}^\dagger\hat{a}$ in the lab frame and express them in term of the expectation values for $\hat{a}$ and $\hat{a}^\dagger\hat{a}$ in the rotating frame. We start with our expression for $\langle\hat{a}^\dagger\hat{a}\rangle$ in the lab frame

$$\langle\hat{a}^\dagger\hat{a}\rangle_{\text{LAB}} = \text{Tr}(\hat{a}^\dagger\hat{a}\hat{\rho}) = \text{Tr}(\hat{a}^\dagger\hat{a}\hat{U}\hat{\chi}\hat{U}^\dagger) \tag{112}$$

and use the cyclic nature of the trace and the relation $\hat{U}^\dagger\hat{a}^\dagger\hat{a}\hat{U} = \hat{a}^\dagger\hat{a}$ to find

$$\langle\hat{a}^\dagger\hat{a}\rangle_{\text{LAB}} = \langle\hat{a}^\dagger\hat{a}\rangle \ . \tag{113}$$

We see that $\langle\hat{a}^\dagger\hat{a}\rangle$ is independent of the frame, as it corresponds to the photon number.
We will now do the same for $\langle\hat{a}\rangle$ and find that $\langle\hat{a}\rangle$ in the lab frame is expressed as

$$\langle\hat{a}\rangle_{\text{LAB}} = \text{Tr}(\hat{a}\hat{\rho}) = \text{Tr}(\hat{a}\hat{U}\hat{\chi}\hat{U}^\dagger) \ , \tag{114}$$

which results in

$$\langle\hat{a}\rangle_{\text{LAB}} = e^{-i\omega_d t}\langle\hat{a}\rangle \ . \tag{115}$$

Where we have once again used the cyclic nature of the trace and the relation $\hat{U}^\dagger\hat{a}\hat{U} = e^{-i\omega_d t}\hat{a}$. To find the long-time solutions for $\langle\hat{a}^\dagger\hat{a}\rangle$ and $\langle\hat{a}\rangle$ in the lab frame, we just substitute the steady state solutions into Eqs. (113) and (115) to find the relations

$$\langle\hat{a}^\dagger\hat{a}\rangle_{LAB,ss} = \langle\hat{a}^\dagger\hat{a}\rangle_{\text{ss}} \tag{116}$$

and

$$\langle\hat{a}\rangle_{LAB,ss} = e^{-i\omega_d t}\langle\hat{a}_{\text{ss}}\rangle \ . \tag{117}$$

D Fourier Transforms

In this section of the appendix, we will derive Eq. (21) and (22). To perform calculations on our system we need an explicit input. We are interested in a laser driven cavity and will therefore investigate the properties and expectation values of our operators for an input field corresponding to a coherent drive with thermal noise, which can be given defined by

$$\hat{b}_{\text{in}}(t) = \langle \hat{b}_{\text{in}} \rangle + \hat{\xi}(t) \ , \tag{118}$$

where $\hat{\xi}(t)$ is the thermal noise, with moments

$$\langle \hat{\xi}(t) \rangle = 0 \tag{119}$$

and

$$\langle \hat{\xi}^\dagger(t_1)\hat{\xi}(t_2) \rangle = n_B \delta(t_1 - t_2). \tag{120}$$

This results in $\langle \hat{b}_{\text{in}}(t) \rangle = \langle \hat{b}_{\text{in}} \rangle$ and $\langle \hat{b}^\dagger_{\text{in}}(t)\, \hat{b}_{\text{in}}(t') \rangle = |\langle \hat{b}_{\text{in}} \rangle|^2 + n_B \delta(t - t')$.

D.1 Evaluating $\langle \hat{a}(t) \rangle$

If we Fourier transform Eq. (7), we get

$$\hat{a}[\omega] = \frac{-\sqrt{\kappa}}{i(\Delta - \omega) + \frac{\kappa}{2}} \, \hat{b}_{\text{in}}[\omega] \ . \tag{121}$$

If we now substitute our input field from Eq. (118), this gives us

$$\hat{b}_{\text{in}}[\omega] = \int_{-\infty}^{\infty} [\langle \hat{b}_{\text{in}} \rangle + \hat{\xi}(t)] e^{i\omega t} \, dt \ = \langle \hat{b}_{\text{in}} \rangle \delta(\omega) + \int_{-\infty}^{\infty} \hat{\xi}(t) e^{i\omega t} \, dt \ . \tag{122}$$

For convenience we define

$$\xi[\omega] = \int_{-\infty}^{\infty} \hat{\xi}(t) e^{i\omega t} \, dt \tag{123}$$

and notice that this has moments $\langle \xi[\omega] \rangle = 0$ and

$$\langle \xi[\omega'] \xi[\omega] \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n_B \delta(t' - t) e^{i\omega t} e^{i\omega' t'} \, dt \, dt' = n_B \delta(\omega + \omega') \ . \tag{124}$$

We can now use Eq. (121), which leads us to

$$\hat{a}[\omega] = \frac{-\sqrt{\kappa}}{i(\Delta - \omega) + \frac{\kappa}{2}} \left(\langle \hat{b}_{\text{in}} \rangle \delta(\omega) + \xi[\omega] \ \right) \tag{125}$$

and

$$\hat{a}(t) = \int_{-\infty}^{\infty} \left(\frac{-\sqrt{\kappa}}{i(\Delta - \omega) + \frac{\kappa}{2}} \left(\langle \hat{b}_{\text{in}} \rangle \delta(\omega) + \xi[\omega] \right) \right) e^{-i\omega t} \, d\omega \quad . \tag{126}$$

From this follows:

$$\langle \hat{a}(t) \rangle = \frac{-\sqrt{\kappa} \langle \hat{b}_{\text{in}} \rangle}{i\Delta + \frac{\kappa}{2}} \ , \tag{127}$$

which agrees with the approach of the master equations as can be seen in Eq. (11).

D.2 Evaluating $\langle \hat{b}_{\text{out}}(t) \rangle$

From Eqs. (111) and (121), follows:

$$\hat{b}_{\text{out}}[\omega] = \frac{i(\Delta - \omega) - \frac{\kappa}{2}}{i(\Delta - \omega) + \frac{\kappa}{2}} \, \hat{b}_{\text{in}}[\omega] \ . \tag{128}$$

We can use $\hat{b}_{\text{in}}[\omega] = \langle \hat{b}_{\text{in}} \rangle \delta(\omega) + \xi[\omega]$ again to find:

$$\hat{b}_{\text{out}}(t) = \langle \hat{b}_{\text{in}} \rangle \frac{i\Delta - \frac{\kappa}{2}}{i\Delta + \frac{\kappa}{2}} + \int_{-\infty}^{\infty} \frac{i(\Delta - \omega) - \frac{\kappa}{2}}{i(\Delta - \omega) + \frac{\kappa}{2}} \xi[\omega] \ . \tag{129}$$

This results in

$$\langle \hat{b}_{\text{out}}(t) \rangle = \langle \hat{b}_{\text{in}} \rangle \frac{i\Delta - \frac{\kappa}{2}}{i\Delta + \frac{\kappa}{2}} \ . \tag{130}$$

D.3 Evaluating $\langle \hat{a}^\dagger(t') \hat{a}(t) \rangle$

By using the same convention of $f^\dagger[\omega] = (f[-\omega])^\dagger$as in [6], we can now express Hermitian conjugate terms to find

$$\hat{a}^\dagger[\omega'] \hat{a}[\omega] = \frac{-\sqrt{\kappa}}{i(\Delta + \omega') + \frac{\kappa}{2}} \frac{-\sqrt{\kappa}}{-i(\Delta - \omega) + \frac{\kappa}{2}} \, \hat{b}^\dagger_{\text{in}}[\omega'] \, \hat{b}_{\text{in}}[\omega] \ , \tag{131}$$

which with our input, can be written as

$$\hat{a}^\dagger[\omega'] \hat{a}[\omega] = \\ \frac{-\sqrt{\kappa}}{i(\Delta + \omega') + \frac{\kappa}{2}} \frac{-\sqrt{\kappa}}{-i(\Delta - \omega) + \frac{\kappa}{2}} \left(\langle \hat{b}^\dagger_{\text{in}} \rangle \delta(-\omega') + \xi^\dagger[-\omega'] \right) \left(\langle \hat{b}_{\text{in}} \rangle \delta(\omega) + \xi[\omega] \right) \tag{132}$$

and equals

$$\hat{a}^\dagger[\omega']\hat{a}[\omega] = \frac{-\sqrt{\kappa}}{i(\Delta+\omega')+\frac{\kappa}{2}}\frac{-\sqrt{\kappa}}{-i(\Delta-\omega)+\frac{\kappa}{2}}\Big(|\langle\hat{b}_{\rm in}\rangle|^2\delta(-\omega')\delta(\omega)+\xi^\dagger[-\omega']\langle\hat{b}_{\rm in}\rangle\delta(\omega) + \xi[\omega]\langle\hat{b}^\dagger_{\rm in}\rangle\delta(-\omega')+\xi^\dagger[-\omega']\xi[\omega]\Big)\ . \tag{133}$$

By transforming back to time-space we find

$$\hat{a}^\dagger(t)\hat{a}(t') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-\sqrt{\kappa}}{i(\Delta+\omega')+\frac{\kappa}{2}}\frac{-\sqrt{\kappa}}{-i(\Delta-\omega)+\frac{\kappa}{2}}\Big(|\langle\hat{b}_{\rm in}\rangle|^2\delta(-\omega')\delta(\omega) + \xi^\dagger[-\omega']\langle\hat{b}_{\rm in}\rangle\delta(\omega)+\xi[\omega]\langle\hat{b}^\dagger_{\rm in}\rangle\delta(-\omega')+\xi^\dagger[-\omega']\xi[\omega]\Big)e^{-i\omega't'}e^{-i\omega t}\,d\omega'\,d\omega\ , \tag{134}$$

resulting in

$$\begin{aligned}\langle\hat{a}^\dagger(t)\hat{a}(t')\rangle &= \frac{\kappa|\langle\hat{b}_{\rm in}\rangle|^2}{\Delta^2+(\frac{\kappa}{2})^2} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{-\sqrt{\kappa}}{i(\Delta+\omega')+\frac{\kappa}{2}}\frac{-\sqrt{\kappa}}{-i(\Delta-\omega)+\frac{\kappa}{2}}n_B\delta(\omega'+\omega)e^{-i\omega't'}e^{-i\omega t}\,d\omega'\,d\omega \\ &= \frac{\kappa|\langle\hat{b}_{\rm in}\rangle|^2}{\Delta^2+(\frac{\kappa}{2})^2} + \int_{-\infty}^{\infty}\frac{\kappa n_B}{(\Delta-\omega)^2+(\frac{\kappa}{2})^2}e^{i\omega(t'-t)}\ d\omega \\ &= \frac{\kappa|\langle\hat{b}_{\rm in}\rangle|^2}{\Delta^2+(\frac{\kappa}{2})^2} + n_B e^{(\frac{-\kappa}{2}+i\Delta)|t'-t|}\ .\end{aligned} \tag{135}$$

For $t=t'$, we get $\langle\hat{a}^\dagger(t)\hat{a}(t')\rangle = \frac{\kappa|\langle\hat{b}_{\rm in}\rangle|^2}{\Delta^2+(\frac{\kappa}{2})^2}+n_B$. Where we have used

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\kappa n_B}{(\Delta-\omega)^2+(\frac{\kappa}{2})^2}e^{i\omega(t'-t)}\ d\omega \\ &= n_B\int_{-\infty}^{\infty}\frac{2\frac{\kappa}{2}}{(\omega)^2+(\frac{\kappa}{2})^2}e^{i(-\omega+\Delta)(t'-t)}\ d\omega \\ &= n_B e^{(\frac{\kappa}{2}+\Delta)|t'-t|}\ .\end{aligned} \tag{136}$$

D.4 Evaluating $\langle\hat{\boldsymbol{b}}^\dagger_{\rm \bf out}(t)\,\hat{\boldsymbol{b}}_{\rm \bf out}(t')\rangle$

Lastly, we look at

$$\hat{b}^\dagger_{\rm out}[\omega']\,\hat{b}_{\rm out}[\omega] = \frac{-i(\Delta+\omega')-\frac{\kappa}{2}}{-i(\Delta+\omega')+\frac{\kappa}{2}}\frac{i(\Delta-\omega)-\frac{\kappa}{2}}{i(\Delta-\omega)+\frac{\kappa}{2}}\,\hat{b}^\dagger_{\rm in}[\omega]\,\hat{b}_{\rm in}[\omega]\ . \tag{137}$$

With our input, this leads to

$$\hat{b}^\dagger_{\rm out}(t)\,\hat{b}_{\rm out}(t') =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-i(\Delta+\omega')-\frac{\kappa}{2}}{-i(\Delta+\omega')+\frac{\kappa}{2}}\frac{i(\Delta-\omega)-\frac{\kappa}{2}}{i(\Delta-\omega)+\frac{\kappa}{2}}\bigg(|\langle\hat{b}_{\rm in}\rangle|^2\delta(-\omega')\delta(\omega)$$
$$+\,\xi^\dagger[-\omega']\langle\hat{b}_{\rm in}\rangle\delta(\omega)+\xi[\omega]\langle\hat{b}^\dagger_{\rm in}\rangle\delta(-\omega')+\xi^\dagger[-\omega']\xi[\omega]\bigg)e^{-i\omega' t'}e^{-i\omega t}\,d\omega'\,d\omega\ , \tag{138}$$

and can be simplified to

$$\langle\hat{b}^\dagger_{\rm out}(t)\,\hat{b}_{\rm out}(t')\rangle = |\langle\hat{b}_{\rm in}\rangle|^2+$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-i(\Delta+\omega')-\frac{\kappa}{2}}{-i(\Delta+\omega')+\frac{\kappa}{2}}\frac{i(\Delta-\omega)-\frac{\kappa}{2}}{i(\Delta-\omega)+\frac{\kappa}{2}}n_B\delta(\omega+\omega')e^{-i\omega' t'}e^{-i\omega t}\,d\omega'\,d\omega\ , \tag{139}$$

which results in

$$\langle\hat{b}^\dagger_{\rm out}(t)\,\hat{b}_{\rm out}(t')\rangle = |\langle\hat{b}_{\rm in}\rangle|^2 + n_B\delta(t-t') = \langle\hat{b}^\dagger_{\rm in}(t)\,\hat{b}_{\rm in}(t)\rangle\ . \tag{140}$$

We see that the amount of photons entering the cavity equals the amount of photons leaving the cavity, implying we are operating the cavity in a steady state with a coherent input drive.

E Transient Regime

To gain a deeper understanding of what happens when an open cavity is driven by a coherent laser, we will explicitly solve the differential Eqs. (9) and (10) found in Section 1.4 for $\langle\hat{a}(t)\rangle$ and $\langle\hat{a}(t)^\dagger\hat{a}(t)\rangle$. This will enable us to obtain the necessary expressions for evaluating the energy flows explicitly and to investigate them in the transient regime. The differential Eqs. (141) and (142) for $\langle\hat{a}(t)\rangle$ and $\langle\hat{a}(t)^\dagger\hat{a}(t)\rangle$ are

$$\partial_t\langle\hat{a}\rangle = (-i\Delta-\frac{\kappa}{2})\langle\hat{a}\rangle - \sqrt{\kappa}\langle\hat{b}_{\rm in}\rangle \tag{141}$$

and

$$\partial_t\langle\hat{a}^\dagger\hat{a}\rangle = -\sqrt{\kappa}\left(\langle\hat{b}^\dagger_{\rm in}\rangle\hat{a}+\langle\hat{b}_{\rm in}\rangle\hat{a}^\dagger\right) - \kappa\langle\hat{a}^\dagger\hat{a}\rangle + \kappa n_B\ . \tag{142}$$

For Eq. (141) we find

$$\langle\hat{a}(t)\rangle = c_1 e^{(-i\Delta-\frac{\kappa}{2})t} + \frac{\sqrt{\kappa}\langle\hat{b}_{\rm in}\rangle}{-i\Delta-\frac{\kappa}{2}}\ , \tag{143}$$

where we can fix the initial state to be the vacuum state; $\langle\hat{a}(t_0)\rangle = 0$, by setting $c_1 = -\frac{\sqrt{\kappa}\langle\hat{b}_{\rm in}\rangle}{-i\Delta-\frac{\kappa}{2}}e^{-(-i\Delta-\frac{\kappa}{2})t_0}$. This gives us

$$\langle \hat{a}(t) \rangle = \frac{\sqrt{\kappa}\langle \hat{b}_{\text{in}} \rangle}{-i\Delta - \frac{\kappa}{2}} \left(1 - e^{(-i\Delta - \frac{\kappa}{2})(t-t_0)}\right) \ . \tag{144}$$

We can verify that $\langle \hat{a}(t) \rangle$ will tend to the steady state by

$$\lim_{t\to\infty} \langle \hat{a}(t) \rangle = \langle \hat{a} \rangle_{\text{ss}} \ . \tag{145}$$

We can now use this result to substitute $\langle \hat{a}(t) \rangle$ into Eq. (141), which gives us

$$\begin{aligned} \partial_t \langle \hat{a}^\dagger \hat{a} \rangle = &- \kappa |\langle \hat{b}_{\text{in}} \rangle|^2 \Bigg(\frac{1}{-i\Delta - \frac{\kappa}{2}} (1 - e^{(-i\Delta - \frac{\kappa}{2})(t-t_0)}) \\ &+ \frac{1}{i\Delta - \frac{\kappa}{2}} (1 - e^{(i\Delta - \frac{\kappa}{2})(t-t_0)}) \Bigg) - \kappa \langle \hat{a}^\dagger \hat{a} \rangle + \kappa n_B \ . \end{aligned} \tag{146}$$

This equation can be simplified to

$$\begin{aligned} \partial_t \langle \hat{a}^\dagger \hat{a} \rangle = &- \frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} \Bigg(- \kappa + e^{-\frac{\kappa}{2}(t-t_0)} (-2\Delta \text{sin}(\Delta(t-t_0)) \\ &+ \kappa \text{cos}(\Delta(t-t_0))) \Bigg) - \kappa \langle \hat{a}^\dagger \hat{a} \rangle + \kappa n_B \end{aligned} \tag{147}$$

and can be solved by

$$\begin{aligned} \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = &e^{-\kappa t} (c_2 + \frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} \int_{t_0}^{t} \kappa + \frac{n_B (\Delta^2 + \frac{\kappa}{2}^2)}{|\langle \hat{b}_{\text{in}} \rangle|^2} e^{\kappa x} \\ &- e^{-\frac{\kappa}{2}(-x-t_0)} (2\Delta \text{sin}(\Delta(x-t_0)) + \kappa \text{cos}(\Delta(x-t_0)))\, dx) \ , \end{aligned} \tag{148}$$

which equals

$$\begin{aligned} \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle = c_2 e^{-\kappa t} + &\frac{2\kappa}{(\Delta^2 + \frac{\kappa}{2}^2)} |\langle \hat{b}_{\text{in}} \rangle|^2 \left(e^{\kappa(t_0 - t)} - e^{\frac{\kappa}{2}(t_0 - t)} \text{cos}(\Delta(t-t_0)) \right) \\ &+ \left(\frac{\kappa |\langle \hat{b}_{\text{in}} \rangle|^2}{\Delta^2 + \frac{\kappa}{2}^2} + n_B \right) (1 - e^{\kappa(t_0 - t)}) \ . \end{aligned} \tag{149}$$

Now we can fix the initial state to be the vacuum state again by setting $c_2 = 0$:

$$\langle \hat{a}^\dagger(t_0) \hat{a}(t_0) \rangle = c_2 e^{-\kappa t_0} = 0 \ , \tag{150}$$

resulting in the final equation

$$
\begin{aligned}
\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle =& \frac{2\kappa|\langle \hat{b}_{\text{in}}\rangle|^2}{\Delta^2+\frac{\kappa}{2}^2}\left(e^{-\kappa(t-t_0)} - e^{-\frac{\kappa}{2}(t-t_0)}\cos(\Delta(t-t_0))\right) \\
&+ \left(\frac{\kappa|\langle \hat{b}_{\text{in}}\rangle|^2}{\Delta^2+\frac{\kappa}{2}^2} + n_B\right)(1-e^{-\kappa(t-t_0)}) \ .
\end{aligned}
\tag{151}
$$

It can again be verified that this tends to the steady state by taking the limit of t to inifinty

$$
\lim_{t\to\infty} \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle_{\text{ss}} \quad . \tag{152}
$$

For $n_B = 0$, this results in

$$
\langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \frac{2\kappa|\langle \hat{b}_{\text{in}}\rangle|^2}{\Delta^2+\frac{\kappa}{2}^2}\left(e^{-\kappa(t-t_0)} - 2e^{-\frac{\kappa}{2}(t-t_0)}\cos(\Delta(t-t_0)) + 1\right) \ . \tag{153}
$$

F Power and Heat

In this section of the appendix we derive the definitions of power and heat motivated by the master equation. We can do this by looking at the inner energy U which we can define as

$$
U = \text{Tr}(\hat{H}\hat{\rho}) \ . \tag{154}
$$

From the inner energy, power and heat current are usually derived by its time-derivative in the following way:

$$
\partial_t U = \underbrace{\text{Tr}(\dot{\hat{H}}\hat{\rho})}_{\text{P}} + \underbrace{\text{Tr}(\hat{H}\dot{\hat{\rho}})}_{\text{J}} = \underbrace{\text{Tr}(\dot{\hat{H}}_d\hat{\rho})}_{\text{P}} + \underbrace{\text{Tr}(\hat{H}_0\dot{\hat{\rho}}) + \text{Tr}(\hat{H}_d\dot{\hat{\rho}})}_{\text{J}} \ , \tag{155}
$$

where the power is obtained from the time derivative of the Hamiltonian and the heat current is obtained from the time derivative of the density matrix. We can evaluate both terms, which results in

$$
\begin{aligned}
\text{Tr}(\dot{\hat{H}}_d\hat{\rho}) &= -\sqrt{\kappa}\omega_d Tr\left((\langle \hat{b}_{\text{in}}^\dagger\rangle e^{i\omega_d t}\hat{a} + \langle \hat{b}_{\text{in}}\rangle e^{-i\omega_d t}\hat{a}^\dagger)\rho\right) \\
&= -\sqrt{\kappa}\omega_d\left(\langle \hat{b}_{\text{in}}^\dagger\rangle\langle \hat{a}\rangle) + \langle \hat{b}_{\text{in}}\rangle\langle \hat{a}^\dagger\rangle\right)
\end{aligned}
\tag{156}
$$

and

$$
\text{Tr}(\hat{H}\dot{\hat{\rho}}) = Tr\Bigg((\Omega\hat{a}^\dagger\hat{a} + i\sqrt{\kappa}(\langle \hat{b}_{\text{in}}^\dagger\rangle e^{i\omega_d t}\hat{a} - \langle \hat{b}_{\text{in}}\rangle e^{-i\omega_d t}\hat{a}^\dagger))\dot{\rho}\Bigg) \ . \tag{157}
$$

To evaluate Eq. (157), we can use the following three expressions:

$$\mathrm{Tr}(\hat{a}^\dagger \hat{a} \dot{\hat{\rho}}) = -\sqrt{\kappa}\left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle\right) + \kappa n_B - \kappa n_B \langle \hat{a}^\dagger \hat{a} \rangle \ , \tag{158}$$

$$\mathrm{Tr}(\hat{a} \dot{\hat{\rho}}) = -i\Omega \langle \hat{a} \rangle_{\mathrm{LAB}} - \sqrt{\kappa} \langle \hat{b}_{\mathrm{in}} \rangle e^{-i\omega_d t} + \frac{1}{2}\kappa \langle \hat{a} \rangle_{\mathrm{LAB}} \ , \tag{159}$$

$$\mathrm{Tr}(\hat{a}^\dagger \dot{\hat{\rho}}) = i\Omega \langle \hat{a}^\dagger \rangle_{\mathrm{LAB}} - \sqrt{\kappa} \langle \hat{b}_{\mathrm{in}}^\dagger \rangle e^{i\omega_d t} + \frac{1}{2}\kappa \langle \hat{a}^\dagger \rangle_{\mathrm{LAB}} \ . \tag{160}$$

With these expressions we can write Eq. (157) as

$$\mathrm{Tr}(\hat{H} \dot{\hat{\rho}}) = \kappa \Omega (n_B - \langle \hat{a}^\dagger \hat{a} \rangle) + \frac{1}{2} i \sqrt{\kappa} \kappa \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle\right) \ . \tag{161}$$

From this follows that the power and the heat current are defined as follows

$$P = -\sqrt{\kappa}\omega_d \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle\right) + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle) \ , \tag{162}$$

$$J = \Omega \kappa (n_B - \langle \hat{a}^\dagger \hat{a} \rangle) + i \frac{\kappa}{2} \sqrt{\kappa} \left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle - \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle\right) \ . \tag{163}$$

This result comes from our definition of the inner energy as $U = \mathrm{Tr}(\hat{H}\hat{\rho})$, however we are mainly interested in a situation where the drive Hamiltonian has a small influence compared to $\hat{H}_0$. We can therefore also look into the case where we define our inner energy as $U = \mathrm{Tr}(\omega_d \hat{a}^\dagger \hat{a} \hat{\rho})$.

$$U = \omega_d \mathrm{Tr}(\hat{a}^\dagger \hat{a} \hat{\rho}) = \omega_d \langle \hat{a}^\dagger \hat{a} \rangle_{\mathrm{LAB}} \tag{164}$$

and therefore leads to:

$$\partial_t U = \mathrm{Tr}(\hat{H}_0 \dot{\hat{\rho}}) = \omega_d \mathrm{Tr}(\hat{a}^\dagger \hat{a} \dot{\hat{\rho}}) = \omega_d \partial_t \langle \hat{a}^\dagger \hat{a} \rangle_{\mathrm{LAB}} = \omega_d \partial_t \langle \hat{a}^\dagger \hat{a} \rangle \ . \tag{165}$$

With Eq. (10) we can now write this as

$$\partial_t U = \omega_d \left(-\sqrt{\kappa}\left(\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle\right) - \kappa \langle \hat{a}^\dagger \hat{a} \rangle + \kappa n_B\right) \ . \tag{166}$$

Because $\partial_t U = P + J$, where P is the power and J is the heat current, we can define the power and heat current from Eq. (27). As κ is our coupling constant to the bath, we will define J from the terms proportional to κ, resulting in

$$J = \omega_d \kappa (n_B - \langle \hat{a}^\dagger \hat{a} \rangle). \tag{167}$$

This leaves us with a definition of the power

$$P = -\sqrt{\kappa}\omega_d (\langle \hat{b}_{\mathrm{in}}^\dagger \rangle \langle \hat{a} \rangle + \langle \hat{b}_{\mathrm{in}} \rangle \langle \hat{a}^\dagger \rangle), \tag{168}$$

which is the exact same result as we found in Eq. (162).

G Calculation for Energy Flows

In this section of the appendix, we will show the calculation to derive Eq. (38) - (43).

G.1 Evaluating $\partial_t\langle\hat{H}_{\text{bath}}\rangle$

As we are looking into power and heat, which are different energy flows, it is also interesting to see how the different energies of our Hamiltonians change. We will therefore take a closer look at the equation of motion for $\langle\hat{H}_{\text{bath}}\rangle$

$$\partial_t\langle\hat{H}_{\text{bath}}\rangle = \langle i[\hat{H},\hat{H}_{\text{bath}}]\rangle = \langle i[\hat{H}_{\text{int}},\hat{H}_{\text{bath}}]\rangle \ . \tag{169}$$

Substituting the Hamiltonians in Eq. (169) gives us

$$\partial_t\langle\hat{H}_{\text{bath}}\rangle = \left\langle \ \left[\sum_{q'} f_{q'}\hat{a}^\dagger\hat{b}_{q'} - f_{q'}^*\hat{a}\hat{b}_{q'}^\dagger \ , \sum_q \omega_q\hat{b}_q^\dagger\hat{b}_q \ \right]\right\rangle \ , \tag{170}$$

which can be simplified to

$$\begin{aligned}\partial_t\langle\hat{H}_{\text{bath}}\rangle &= \left\langle\sum_q \omega_q\left[f_q\hat{a}^\dagger\hat{b}_q - f_q^*\hat{a}\hat{b}_q^\dagger, \hat{b}_q^\dagger\hat{b}_q\right]\right\rangle \\ &= \left\langle\sum_q \omega_q\left(f_q\hat{a}^\dagger\hat{b}_q + f_q^*\hat{a}\hat{b}_q^\dagger\right)\right\rangle \ .\end{aligned} \tag{171}$$

G.2 Evaluating $\partial_t\langle\hat{H}_{\text{int}}\rangle$

We will also take a closer look at the equation of motion for $\langle\hat{H}_{\text{int}}\rangle$, which is given by

$$\partial_t\langle\hat{H}_{\text{int}}\rangle = \langle i[\hat{H}_{\text{bath}}+\hat{H}_{\text{sys}},\hat{H}_{\text{int}}]\rangle = \langle i[\hat{H}_{\text{sys}},\hat{H}_{\text{int}}]\rangle - \partial_t\langle\hat{H}_{\text{bath}}\rangle \ , \tag{172}$$

where we can see that

$$\langle i[\hat{H}_{\text{sys}},\hat{H}_{\text{int}}]\rangle = \left\langle\Omega\sum_q\left[f_q\hat{a}^\dagger\hat{b}_q + f_q^*\hat{a}\hat{b}_q^\dagger\right]\right\rangle \ , \tag{173}$$

which just equals $\frac{\Omega}{\omega_d}\partial_t\langle\hat{H}_{\text{bath}}\rangle$, resulting in a final equation of motion for $\langle\hat{H}_{\text{int}}\rangle$, which is very similar to Eq. (171)

$$\partial_t\langle\hat{H}_{\text{int}}\rangle = \left\langle\Delta\sum_q[f_q\hat{a}^\dagger\hat{b}_q + f_q^*\hat{a}\hat{b}_q^\dagger]\right\rangle \ . \tag{174}$$

G.3 Evaluating $\partial_t \langle \hat{H}_{\text{sys}} \rangle$

Lastly, we will also take a closer look at the equation of motion for $\langle \hat{H}_{\text{sys}} \rangle$, which is given by

$$\partial_t \langle \hat{H}_{\text{sys}} \rangle = \langle i[\hat{H}_{\text{int}}, \hat{H}_{\text{sys}}] \rangle = - \left\langle \Omega \sum_q [f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a} \hat{b}_q^\dagger] \right\rangle \ , \tag{175}$$

and makes sure that $\partial_t \langle \hat{H} \rangle = 0$:

G.4 Expressing $\partial_t \langle \hat{H}_{\text{sys}} \rangle$, $\partial_t \langle \hat{H}_{\text{int}} \rangle$ and $\partial_t \langle \hat{H}_{\text{bath}} \rangle$

We have found that the equations of motion for all three Hamiltonians are very similar. Namely;

$$\partial_t \langle \hat{H}_{\text{sys}} \rangle = - \left\langle \Omega \sum_q \left[f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a} \hat{b}_q^\dagger \right] \right\rangle \ , \tag{176}$$

$$\partial_t \langle \hat{H}_{\text{int}} \rangle = \left\langle \Delta \sum_q \left[f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a} \hat{b}_q^\dagger \right] \right\rangle \ , \tag{177}$$

$$\partial_t \langle \hat{H}_{\text{bath}} \rangle = \left\langle \omega_d \sum_q \left[f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a} \hat{b}_q^\dagger \right] \right\rangle \ , \tag{178}$$

where we have made the assumption that $\omega_q \approx \omega_d$. To further evaluate these expressions, we need to express $\sum_q \left[f_q \hat{b}_q(t) \right]$ in terms of the operators we are working with; $\hat{b}_{\text{in}}(t)$, $\hat{b}_{\text{out}}(t)$ and $\hat{a}(t)$. By substituting the explicit expression for our bath variables from Eq. (92), we find

$$\sum_q \left[\hat{b}_q(t) \right] = \sum_q \left[e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0) + \int_{t_0}^t e^{-i\omega_q(t-\tau)} f_q^* \hat{a}(\tau)\, d\tau \right] . \tag{179}$$

We see that the first term in this expression resembles the input field, so we can use Eq. (106), to simplify

$$\sum_q \left[\hat{b}_q(t) \right] = \frac{\sqrt{\kappa}}{f} \hat{b}_{\text{in}}(t) + \sum_q \left[\int_{t_0}^t e^{-i\omega_q(t-\tau)} f_q^* \hat{a}(\tau)\, d\tau \right] . \tag{180}$$

This results in

$$\sum_q \left[f_q \hat{b}_q(t) \right] = \sqrt{\kappa}\, \hat{b}_{\text{in}}(t) + \sum_q |f_q|^2 \int_{t_0}^t e^{-i\omega_q(t-\tau)} \hat{a}(\tau)\, d\tau \ . \tag{181}$$

Now we can use Eq. (101) again to find

$$\sum_q \left[f_q \hat{b}_q(t) \right] = \sqrt{\kappa}\, \hat{b}_{\rm in}(t) + \int_{t_0}^{t} \kappa\delta(t-\tau)\hat{a}(\tau)\, d\tau, \tag{182}$$

which, with Eq, (103) simplifies even further to

$$\sum_q \left[f_q \hat{b}_q(t) \right] = \sqrt{\kappa}\, \hat{b}_{\rm in}(t) + \frac{\kappa}{2}\hat{a}(t) \ . \tag{183}$$

Now that we have found an explicit expression for $\sum_q \left[f_q \hat{b}_q(t) \right]$, we can substitute this back into $\langle \sum_q [f_q \hat{a}^\dagger(t)\hat{b}_q(t) + f_q^* \hat{a}(t)\hat{b}_q^\dagger(t)]\rangle$, and find

$$\left\langle \sum_q [f_q \hat{a}^\dagger \hat{b}_q + f_q^* \hat{a}\hat{b}_q^\dagger] \right\rangle = \sqrt{\kappa}\left(\langle \hat{b}_{\rm in}^\dagger\, \hat{a}\rangle + \langle \hat{a}^\dagger\, \hat{b}_{\rm in}\rangle \right) + \kappa \langle \hat{a}^\dagger \hat{a}\rangle \ . \tag{184}$$

Here we can use the input output-relation from Eq. (30) again and write the right hand side of this equation in terms of the input and output field to find:

$$\begin{aligned} \partial_t \langle \hat{H}_{\rm bath}\rangle = \omega_d \Big(&\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm out}\rangle - \langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle \\ +\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm in}\rangle - \langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle + &\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm out}\rangle - \langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm in}\rangle - \langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm out}\rangle + \langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle \Big) \\ = \omega_d &\left(\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm out}\rangle - \langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle \right) \ . \end{aligned} \tag{185}$$

H Evaluating Covariances of the Output

To evaluate the expressions $\langle\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm in}\rangle\rangle$, $\langle\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm out}\rangle\rangle$ and $\langle\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm out}\rangle\rangle$, we use the input output relation

$$\hat{b}_{\rm out} = \hat{b}_{\rm in} + \sqrt{\kappa}\hat{a} \tag{186}$$

and substitute this in to find

$$\langle\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm in}\rangle\rangle = \langle\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle\rangle + \sqrt{\kappa}\langle\langle \hat{a}^\dagger\, \hat{b}_{\rm in}\rangle\rangle \ , \tag{187}$$

$$\langle\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm out}\rangle\rangle = \langle\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle\rangle + \sqrt{\kappa}\langle\langle \hat{b}_{\rm in}^\dagger\, \hat{a}\rangle\rangle \tag{188}$$

and

$$\langle\langle \hat{b}_{\rm out}^\dagger\, \hat{b}_{\rm out}\rangle\rangle = \langle\langle \hat{b}_{\rm in}^\dagger\, \hat{b}_{\rm in}\rangle\rangle + \sqrt{\kappa}(\langle\langle \hat{b}_{\rm in}^\dagger\, \hat{a}\rangle\rangle + \langle\langle \hat{a}^\dagger\, \hat{b}_{\rm in}\rangle\rangle) + \kappa\langle\langle \hat{a}^\dagger \hat{a}\rangle\rangle \ . \tag{189}$$

Now we can evaluate $\langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{a}\rangle\rangle$ and $\langle\langle \hat{a}^\dagger\, \hat{b}_{\text{in}}\rangle\rangle$ according to [11], where we can find

$$\langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{a}\rangle\rangle = \langle\langle \hat{a}^\dagger\, \hat{b}_{\text{in}}\rangle\rangle = -\sqrt{\kappa}\frac{n_B}{2} \ . \tag{190}$$

Therefore, we can express

$$\langle\langle \hat{b}^\dagger_{\text{out}}\, \hat{b}_{\text{in}}\rangle\rangle = \langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{b}_{\text{in}}\rangle\rangle - \frac{\kappa}{2}n_B \ , \tag{191}$$

$$\langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{b}_{\text{out}}\rangle\rangle = \langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{b}_{\text{in}}\rangle\rangle - \frac{\kappa}{2}n_B \ , \tag{192}$$

which in turn gives us the covariance of the output

$$\langle\langle \hat{b}^\dagger_{\text{out}}\, \hat{b}_{\text{out}}\rangle\rangle = \langle\langle \hat{b}^\dagger_{\text{in}}\, \hat{b}_{\text{in}}\rangle\rangle + \kappa\left[\langle\langle \hat{a}^\dagger \hat{a}\rangle\rangle - n_B\right] \ . \tag{193}$$

I Input Output Theory in the Rotating Frame

In this part of the appendix we show how input output theory can be applied for calculations in the rotating frame. To switch into the rotating frame in Input Output Theory, we use the unitary operator $U = e^{i\omega_d t \hat{a}^\dagger \hat{a} + \sum_q i\omega_d t \hat{b}^\dagger_q \hat{b}_q}$. We can apply the same transformation as we did in for the Master equation, resulting in

$$\hat{\tilde{H}} = \hat{U}^\dagger \hat{H}_{\text{sys}} \hat{U} + \hat{U}^\dagger \hat{H}_{\text{bath}} \hat{U} + \hat{U}^\dagger \hat{H}_{\text{int}} \hat{U} + i\dot{\hat{U}}^\dagger \hat{U}. \tag{194}$$

This results in:

$$\hat{\tilde{H}} = \hat{H}_{\text{sys}} + \hat{H}_{\text{bath}} + \hat{U}^\dagger \hat{H}_{\text{int}} \hat{U} - \omega_d \hat{a}^\dagger \hat{a} \ . \tag{195}$$

Which means that $\hat{H}_{\text{sys}} = \Omega \hat{a}^\dagger \hat{a}$, will become $\Delta \hat{a}^\dagger \hat{a}$. And we find $\hat{U}^\dagger \hat{H}_{\text{int}} \hat{U} = -i\sum_q [e^{i\omega_d t - \omega_q} f_q \hat{a}^\dagger \hat{b}_q - e^{-i\omega_d t + \omega_q} f^*_q \hat{a} \hat{b}^\dagger_q]$. This results in

$$\hat{b}_{\text{q,rot}}(t) = e^{-i\omega_q(t-t_0)}\hat{b}_q(t_0) + \int_{t_0}^{t} e^{(-i\omega_d t + \omega_q)} e^{-i\omega_q(t-\tau)} f^*_q \hat{a}(\tau)\, d\tau \tag{196}$$

and

$$\partial_t \hat{a}_{\text{rot}} = i[\hat{H}_{\text{sys}}, \hat{a}_{\text{rot}}] - \sum_q e^{i\omega_d t - \omega_q} f_q\ \hat{b}_{\text{q,rot}} \ . \tag{197}$$

Which we, in turn, can write as

$$\partial_t \hat{a}_{\text{rot}} = i[\hat{H}_{\text{sys}}, \hat{a}_{\text{rot}}] - \frac{\kappa}{2}\hat{a} - \sum_q f_q e^{(i\omega_d t - \omega_q)} e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0) \ . \tag{198}$$

and gives us

$$\hat{b}_{\text{in,rot}} = \frac{1}{\sqrt{2\pi\rho}} \sum_q e^{(i\omega_d t - \omega_q)} e^{-i\omega_q(t-t_0)} \hat{b}_q(t_0) \ = e^{(i\omega_d t - \omega_q)}\ \hat{b}_{\text{in}} \ , \tag{199}$$

$$\hat{b}_{\text{out,rot}} = e^{(i\omega_d t - \omega_q)}\ \hat{b}_{\text{out}} \ . \tag{200}$$

We still find that

$$\partial_t \hat{a}_{,\mathrm{rot}} = i[\hat{H}_{\mathrm{sys}}, \hat{a}_{,\mathrm{rot}}] + \frac{\kappa}{2}\hat{a}_{,\mathrm{rot}} - \sqrt{\kappa}\,\hat{b}_{\mathrm{out,rot}}(t) \ , \tag{201}$$

$$\partial_t \hat{a}_{,\mathrm{rot}} = i[\hat{H}_{\mathrm{sys}}, \hat{a}_{,\mathrm{rot}}] - \frac{\kappa}{2}\hat{a}_{,\mathrm{rot}} - \sqrt{\kappa}\,\hat{b}_{\mathrm{in,rot}}(t) \tag{202}$$

and

$$\boxed{\hat{b}_{\mathrm{out,rot}} = \hat{b}_{\mathrm{in,rot}} + \sqrt{\kappa}\hat{a}_{,\mathrm{rot}}(t)} \ . \tag{203}$$

J Entropy Production for Gaussian States

In this part of the appendix we show the calculations to derive Eq. (68) and (73). We start with the entropy production rate (67), here reproduced for convenience.

$$\partial_t \Sigma = \frac{-J}{T} + k_B \partial_t S_{\mathrm{vN}}[\hat{\rho}] \ , \tag{204}$$

where J is our heat current, T is the temperature, k_B is the Boltzmann constant, S_{vN} the von Neumann entropy and $\hat{\rho}$ the density matrix of the system. For a Gaussian state we can express the entropy as

$$k_B S_{\mathrm{vN}}[\hat{\rho}] = k_B \left[(\sigma+1)\ln(\sigma+1) - \sigma\ln(\sigma)\right] \ , \tag{205}$$

as is demonstrated in [9], where σ is defined as

$$\sigma = \sqrt{\alpha\beta - \gamma^2} - \frac{1}{2} \ , \tag{206}$$

with $\alpha = i\frac{1}{2}\langle\langle(\hat{a}^\dagger - \hat{a})^2\rangle\rangle$, $\beta = \frac{1}{2}\langle\langle(\hat{a}^\dagger + \hat{a})^2\rangle\rangle$ and $\gamma = i\frac{1}{4}\langle\langle(\hat{a}^\dagger - \hat{a})(\hat{a}^\dagger + \hat{a}) + (\hat{a}^\dagger + \hat{a})(\hat{a}^\dagger - \hat{a})\rangle\rangle$. We can use this definition to find

$$\sigma = \sqrt{\langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle^2 + \langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle - \langle\langle\hat{a}^\dagger\hat{a}^\dagger\rangle\rangle\langle\langle\hat{a}\hat{a}\rangle\rangle + \frac{1}{4}} - \frac{1}{2} \ . \tag{207}$$

If we now assume no squeezing ($\langle\langle\hat{a}^\dagger\hat{a}^\dagger\rangle\rangle = \langle\langle\hat{a}\hat{a}\rangle\rangle = 0$), we can write this expression as

$$\sigma = \sqrt{(\langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle + \frac{1}{2})^2} - \frac{1}{2} = \langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle \ . \tag{208}$$

We can now calculate its time derivative as follows

$$\partial_t \sigma = \partial_t\langle\hat{a}^\dagger\hat{a}\rangle \ - \langle\hat{a}^\dagger\rangle\partial_t\langle\hat{a}\rangle - \langle\hat{a}\rangle\partial_t\langle\hat{a}^\dagger\rangle = \kappa(n_B - \langle\langle\hat{a}^\dagger\hat{a}\rangle\rangle) \ . \tag{209}$$

When we take the time derivative of the von Neumann entropy, we get

$$\partial_t k_B S_{\mathrm{vN}}[\hat{\rho}] = k_B \partial_t \sigma(\ln(\sigma+1) - \ln(\sigma)) \ , \tag{210}$$

which, by substituting in σ and $\partial_t\sigma$ results in

$$\partial_t k_B S_{\mathrm{vN}}[\hat\rho] = k_B\kappa(n_B - \langle\langle\hat a^\dagger\hat a\rangle\rangle)\ln\left(\frac{\langle\langle\hat a^\dagger\hat a\rangle\rangle + 1}{\langle\langle\hat a^\dagger\hat a\rangle\rangle}\right) . \tag{211}$$

Now we use the Bose Einstein distribution

$$n_B = \frac{1}{e^{\omega_d/k_B T} - 1} , \tag{212}$$

to write the temperature as

$$\frac{1}{T} = \frac{k_B}{\omega_d}\ln\left(\frac{n_B + 1}{n_B}\right) \tag{213}$$

by using $\epsilon = \omega_d$. Now we can substitute $\frac{-J}{T}$ and Eq. (211) into Eq. (204), to find

$$\begin{aligned}
\partial_t\Sigma =& -\kappa(n_B - \langle\hat a^\dagger\hat a\rangle)k_B\ln\left(\frac{n_B+1}{n_B}\right) \qquad (214)\\
&+ k_B(n_B - \langle\langle\hat a^\dagger\hat a\rangle\rangle)\ln\left(\frac{\langle\langle\hat a^\dagger\hat a\rangle\rangle + 1}{\langle\langle\hat a^\dagger\hat a\rangle\rangle}\right)\\
=& k_B\kappa\Bigg(-(n_B - \langle\hat a^\dagger\hat a\rangle)\ln\big(\frac{n_B+1}{n_B}\big) \qquad (215)\\
&+ (n_B - \langle\langle\hat a^\dagger\hat a\rangle\rangle)\ln\big(\frac{\langle\langle\hat a^\dagger\hat a\rangle\rangle + 1}{\langle\langle\hat a^\dagger\hat a\rangle\rangle}\big)\Bigg)\\
=& k_B\kappa(n_B - \langle\langle\hat a^\dagger\hat a\rangle\rangle)\Bigg(\ln\big(\frac{\langle\langle\hat a^\dagger\hat a\rangle\rangle + 1}{\langle\langle\hat a^\dagger\hat a\rangle\rangle}\big) - \ln\big(\frac{n_B+1}{n_B}\big)\Bigg) \qquad (216)\\
&+ k_B\kappa\langle\hat a^\dagger\rangle\langle\hat a\rangle\ln\big(\frac{n_B+1}{n_B}\big)
\end{aligned}$$

If we now want to do the same calculation for J' instead of J, we simply use $J' = J + \omega_d\kappa\langle\hat a^\dagger\rangle\langle\hat a\rangle$, which results in

$$\begin{aligned}
\partial_t\Sigma' =& \partial_t\Sigma - k_B\kappa\langle\hat a^\dagger\rangle\langle\hat a\rangle\ln\left(\frac{n_B+1}{n_B}\right)\\
=& k_B\kappa(n_B - \langle\langle\hat a^\dagger\hat a\rangle\rangle)\Bigg(\ln\big(\frac{\langle\langle\hat a^\dagger\hat a\rangle\rangle + 1}{\langle\langle\hat a^\dagger\hat a\rangle\rangle}\big) - \ln\big(\frac{n_B+1}{n_B}\big)\Bigg) .
\end{aligned} \tag{217}$$

References

[1] W. Niedenzu, M. Huber, and E. Boukobza. Concepts of work in autonomous quantum heat engines. *Quantum*, 3:195, October 2019. `https://doi.org/10.22331/q-2019-10-14-195`.

[2] S. Vinjanampathy and J. Anders. Quantum thermodynamics. *Contemporary Physics*, 57(4):545–579, 2016. `https://doi.org/10.1080/00107514.2016.1201896`.

[3] Patrick P. Potts. Quantum thermodynamics, 2024. `https://arxiv.org/abs/2406.19206`.

[4] J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffèves. The energetic cost of work extraction. *Phys. Rev. Lett.*, 124:130601, Mar 2020. `https://link.aps.org/doi/10.1103/PhysRevLett.124.130601`.

[5] C. Gardiner and P. Zoller. *Quantum Noise*. Springer Berlin, Heidelberg, 2010.

[6] A. A. Clerk, M. H. Devoret, S. M. Girvin, Florian Marquardt, and R. J. Schoelkopf. Introduction to quantum noise, measurement, and amplification. *Rev. of modern Phys.*, 82(2):1155–1208, 4 2010. `https://doi.org/10.1103/revmodphys.82.1155`.

[7] S. Vinjanampathy and J. Anders. Quantum thermodynamics. *Contemporary Physics*, 57(4):545–579, July 2016. `http://dx.doi.org/10.1080/00107514.2016.1201896`.

[8] P Potts, A. A. S. Kalaee, and A Wacker. A thermodynamically consistent markovian master equation beyond the secular approximation. *New J. Phys.*, 23(12):123013, December 2021. `http://dx.doi.org/10.1088/1367-2630/ac3b2f`.

[9] G. S. Agarwal. Entropy, the wigner distribution function, and the approach to equilibrium of a system of coupled harmonic oscillators. *Phys. Rev. A*, 3:828–831, Feb 1971. `https://link.aps.org/doi/10.1103/PhysRevA.3.828`.

[10] A. A. S. Kalaee, A. Wacker, and P. Potts. Violating the thermodynamic uncertainty relation in the three-level maser. *Phys. Rev. E*, 104(1), July 2021. `http://dx.doi.org/10.1103/PhysRevE.104.L012103`.

[11] C. W. Gardiner and M. J. Collett. Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Phys. Rev. A*, 31:3761–3774, Jun 1985. `https://link.aps.org/doi/10.1103/PhysRevA.31.3761`.