A Keldysh Path Integral Approach to Input–Output Theory

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Master's Thesis

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Abstract

Input–output theory is a well–known tool in cavity electrodynamics and ubiquitous in the description of quantum systems interacting with the environment. In this master's thesis, we present a new approach to input–output theory using the Keldysh path integral formalism in order to get perturbative results for non–linear systems not solvable through the standard methods of input–output theory. To this end we reproduce the standard approach to input–output theory, we give an overview over the Keldysh path integral formalism, and we detail the derivation of the novel Keldysh path integral approach to input–output theory. Subsequently, we apply this approach to a simple system solvable through standard input–output theory and then treat a non–linear system to showcase the specific strength of our approach to yield perturbative results for non–linear systems.

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Introduction

We see the world around us through light that is reflected from the objects in our vicinity and carries the information about their shape and colour to our eyes. Even though the objects under consideration in modern quantum mechanics are no longer visible to the naked eye, many of them are probed through light signals and we gain information about them through the way they interact with this light impinging on them. Nitrogen vacancy centers, mechanical resonators, quantum dots, quantum bits or a single light mode in an optical cavity – all these systems are probed by light signals and their reaction to that light signal gives us access to the physics happening inside them. Originally derived in the context of transmission lines for electrical signals, the theory of how a system reacts to an incoming signal from its environment, referred to as input–output theory in the jargon of the subject, is therefore of general interest to scientists trying to probe the most diverse kinds of quantum systems.

Standard input–output theory treats the systems mentioned above as coupled to a bath of harmonic oscillators which provides an input to the system and into which the system answers with an output. The standard formalism of input–output theory has been widely applicable to quantum systems coupled to the electromagnetic field - the prime example for a field whose normal modes are modelled as harmonic oscillators. The formalism however struggles in dealing with systems that exhibit non-linearities in their own system modes. To extend the formalism in a way that one can effectively derive perturbative results for such cases, we suggest a new approach to input–output theory employing path integral methods from quantum field theory. By framing the scenario of input–output theory in the language of non–equilibrium quantum field theory, a rich toolbox of perturbative techniques becomes accessible and previously unsolvable problems could become manageable. In this thesis, we will present this approach to input–output theory based on the Keldysh path integral and showcase its use on two examples - one solvable through the conventional methods and one where perturbation theory has to be employed.

The thesis is split into three chapters. We firstly introduce the theoretical background to our topic, namely the standard approach to input–output theory in Sec. 1.2 and the Keldysh path integral in Sec. 1.3, and subsequently detail the derivation of our novel approach in Sec. 1.4 in the first chapter. We then move on to apply the new formalism to the damped harmonic oscillator in the second chapter to elucidate our methods and compare our approach to the standard input–output theory and afterwards treat the Kerr oscillator in the third chapter where we will derive perturbative results for the output field of said system. Throughout the whole manuscript, certain technical steps in the calculations will be omitted for the sake of readability, those that are still crucial to the replicability of our derivations can be found in the appendix. Some theoretical prerequisites have also been omitted from the main text and can be found in the appendix. Furthermore certain equations will be boxed to highlight their overall importance.

Contents

Chapter 1

Theoretical Formalism

In this chapter we derive the standard formalism of input–output theory as it is presented in standard references on the topic such as [4] and outline the basic principles of the Keldysh path integral formalism where we closely follow the introductory chapters of [7]. We then synthesize these two approaches into the Keldysh path integral approach to input–output theory. This novel approach builds on previous work by P. Potts.

1.1 General Setting

The physical setting we describe consists of a system S coupled to a bath of harmonic oscillators B via the rotating–wave interaction V. The Hamiltonian of the complete setting is thus given as follows,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{V}, \tag{1.1}$$

where the bosonic bath is described by the following Hamiltonian,

$$\hat{H}_B = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k, \quad \left[\hat{b}_k, \hat{b}_{k'}^\dagger\right] = \delta_{k,k'}, \tag{1.2}$$

and the coupling through the following term,

$$\hat{V} = \sum_k \left(g_k \hat{a}^\dagger \hat{b}_k + g_k \hat{b}_k^\dagger \hat{a}\right). \tag{1.3}$$

The operator $\hat{a}$ describes a mode in the system here, the system Hamiltonian $\hat{H}_S$ is left unspecified.

1.2 Standard Input–Output Theory

Standard input–output theory describes the system and bath modes of the general setting mentioned above through the use of Heisenberg equations. We start here by deriving the time-dependence of a single bath mode,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_k = i\Big[\hat{H},\hat{b}_k\Big] &= i\sum_l \Big[\omega_k \hat{b}_l^\dagger \hat{b}_l + \Big(g_l \hat{a}^\dagger \hat{b}_k + g_k \hat{b}_k^\dagger \hat{a}\Big), \hat{b}_k\Big] \\
&= -i\sum_l \Big(\delta_{l,k}\hat{b}_l + g_l\,\delta_{l,k}\hat{a}\Big) = -i(\omega_k\hat{b}_k + g_k\hat{a}),
\end{aligned} \tag{1.4}
$$

this differential equation yields the following solution (for details see A.1.1),

$$
\begin{aligned}
\hat{b}_k(t) &= e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) - ig_k\int_{t_0}^{t} d\tau e^{-i\omega_k(t-\tau)}\hat{a}(\tau) \\
&= e^{-i\omega_k(t-t_N)}\hat{b}_k(t_N) - ig_k\int_{t_N}^{t} d\tau e^{-i\omega_k(t-\tau)}\hat{a}(\tau),
\end{aligned} \tag{1.5}
$$

where we introduced the times t_0, t_N which are supposed to lie in the far past, far future respectively.

Now we turn to the time-dependence of the system mode $\hat{a}$,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = i\Big[\hat{H}_S + \hat{H}_B + \hat{V}, \hat{a}\Big] = i\Big[\hat{H}_S, \hat{a}\Big] + i\Big[\hat{H}_B, \hat{a}\Big] + i\Big[\hat{V}, \hat{a}\Big]. \tag{1.6}
$$

The second term in the equation above vanishes due to the following commutation relations,

$$
\Big[\hat{b}_k, \hat{a}\Big] = \Big[\hat{b}_k^\dagger, \hat{a}\Big] = 0.
$$

We therefore focus on the third term,

$$
\begin{aligned}
i\Big[\hat{V},\hat{a}\Big] &= i\sum_k \Big[g_k \hat{a}^\dagger \hat{b}_k + g_k \hat{b}_k^\dagger \hat{a}, \hat{a}\Big] = -i\sum_k g_k \hat{b}_k \\
&= -\sum_k \Big(ig_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) + |g_k|^2\int_{t_0}^{t} d\tau e^{i\omega_k(t-\tau)}\hat{a}(\tau)\Big) \\
&= -\sqrt{\kappa}\frac{1}{\sqrt{\kappa}}\sum_k ig_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0) - \sum_k |g_k|^2\int_{t_0}^{t} d\tau e^{i\omega_k(t-\tau)}\hat{a}(\tau),
\end{aligned} \tag{1.7}
$$

where we introduced the coupling constant κ in the last step. We further introduce the input and output operators,

$$
\hat{b}_{\mathrm{in}}(\mathrm{t}) = \frac{1}{\sqrt{\kappa}}\sum_k ig_k e^{-i\omega_k(t-t_0)}\hat{b}_k(t_0), \qquad \hat{b}_{\mathrm{out}}(\mathrm{t}) = \frac{1}{\sqrt{\kappa}}\sum_k ig_k e^{i\omega_k(t_N-t)}\hat{b}_k(t_N), \tag{1.8}
$$

as well as the bath spectral density,

$$
\rho(\omega) = \sum_k |g_k|^2\delta(\omega - \omega_k). \tag{1.9}
$$

Using the last two quantities, we can express Eq. (1.7) as follows,

$$
i\Big[\hat{V},\hat{a}\Big] = -\sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t}) - \int_{t_0}^{t} d\tau \int_0^{\infty} d\omega\,\rho(\omega)e^{-i\omega(t-\tau)}\hat{a}(\tau). \tag{1.10}
$$

With these simplifications the equation of motion of the bath mode $\hat{a}$ is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = i\left[\hat{H}_S, \hat{a}\right] - \sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t}) - \int_{t_0}^{t} d\tau \int_0^{\infty} d\omega\, \rho(\omega) e^{-i\omega(t-\tau)}\hat{a}(\tau). \tag{1.11}$$

Analogously we find,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = i\left[\hat{H}_S, \hat{a}\right] - \sqrt{\kappa}\,\hat{b}_{\mathrm{out}}(\mathrm{t}) + \int_{t}^{t_N} d\tau \int_0^{\infty} d\omega\, \rho(\omega) e^{-i\omega(t-\tau)}\hat{a}(\tau). \tag{1.12}$$

Even though at this stage our equations only contain the input and output operators, we can retrieve the bath modes at the times t_0 and t_N from the input–output operators,

$$\hat{b}_k(t_0) = \frac{\sqrt{\kappa}}{2\pi i g_k}\int_{\omega_k-\delta\omega}^{\omega_k+\delta\omega} d\omega \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t_0)}\,\hat{b}_{\mathrm{in}}(\mathrm{t}), \tag{1.13}$$

$$\hat{b}_k(t_N) = \frac{\sqrt{\kappa}}{2\pi i g_k}\int_{\omega_k-\delta\omega}^{\omega_k+\delta\omega} d\omega \int_{-\infty}^{\infty} dt\, e^{-i\omega(t_N-t)}\,\hat{b}_{\mathrm{out}}(\mathrm{t}), \tag{1.14}$$

for a proof of this relation see A.1.2.

From the equations of motion for the system mode $\hat{a}$, Eqs. (1.12) & (1.11), we get the following relation between the input and output operators,

$$\hat{b}_{\mathrm{out}}(\mathrm{t}) = \hat{b}_{\mathrm{in}}(\mathrm{t}) + \int_{t_0}^{t_N} d\tau \int_0^{\infty} d\omega\, \rho(\omega) e^{-i\omega(t-\tau)}\hat{a}(\tau), \tag{1.15}$$

and we can derive their commutation relations,

$$\left[\hat{b}_{\mathrm{in}}(\mathrm{t}), \hat{b}^{\dagger}_{\mathrm{in}}(t')\right] = \left[\hat{b}_{\mathrm{out}}(\mathrm{t}), \hat{b}^{\dagger}_{\mathrm{out}}(t')\right] = \frac{1}{\kappa}\int_0^{\infty} d\omega\, \rho(\omega) e^{-i\omega(t-t')}. \tag{1.16}$$

1.2.1 Markov Approximation

So far we treated time as a continuous quantity and made no assumptions on the spectrum of the bath. Now we make the following assumptions:

- The bath consists of N equidistantly spaced bath modes, i.e. $\omega_k = \omega_0 + k\delta\omega$ for $k \in \mathbb{N}$.
- Time is discretized such that $t_j = t_0 + j\delta t$ with $j = 0, 1, 2, \ldots, N$ and $t_N = t_{N-1} = t_0 + (N-1)\delta t$.
- The timestep of the discretization is connected to the discretization of the bath modes such that $\delta t \delta\omega = \frac{2\pi}{N}$.

With these assumptions, the bath modes expressed through the input and output operators from Eq. (1.8) can be written as follows,

$$\hat{b}_k(t_0) = \frac{\sqrt{\kappa}}{i g_k}\sum_{j=0}^{2N-1} e^{i\omega_k(t_j-t_0)}\hat{b}_{\mathrm{in}}(t_j), \tag{1.17}$$

$$\hat{b}_k(t_N) = \frac{\sqrt{\kappa}}{i g_k}\sum_{j=0}^{2N-1} e^{-i\omega_k(t_N-t_j)}\hat{b}_{\mathrm{out}}(t_j). \tag{1.18}$$

Additionally to the discretization outlined above, we now make the following two approximations:

- The Fourier transform of $\hat{a}(\tau)$ is peaked at a specific frequency Ω, where the peak has a fixed width γ.
- The spectral density $\rho(\omega)$ is flat in frequency and around Ω can be approximated as follows (Markov approximation),

$$\rho(\omega) \approx \rho(\Omega) \quad \forall \omega \in (\Omega - \gamma, \Omega + \gamma). \tag{1.19}$$

These assumptions let us simplify the third term in the relation between input and output operators from Eq. (1.15) as follows,

$$\begin{aligned} \int_{t_0}^{t_N} d\tau \int_0^{1} d\omega\, \rho(\omega) e^{\ \ i\omega(t\ \ \tau)} \hat{a}(\tau) &\approx \rho(\Omega) \int_{t_0}^{t_N} d\tau \int_{\ 1}^{1} d\omega e^{\ \ i\omega(t\ \ \tau)} \hat{a}(\tau) \\ &= \rho(\Omega) \int_{t_0}^{t_N} 2\pi\delta(t-\tau)\hat{a}(\tau) = \underbrace{2\pi\rho(\Omega)}_{=:\kappa} \hat{a}(t), \end{aligned} \tag{1.20}$$

where we extended the range of integration in frequency in the first step to employ the required delta distribution correlations. The extension of the integration range is justified in [4] by the argument that only nearly resonant frequencies are important here and the nonphysical negative frequencies therefore contribute very little. Equivalently, using these approximations and the relation,

$$\int_{\ 1}^{x_0} dx \delta(x - x_0) = \frac{1}{2}, \tag{1.21}$$

we find,

$$\int_{t_0}^{t} d\tau \int_0^{1} d\omega\, \rho(\omega) e^{\ \ i\omega(t\ \ \tau)} \hat{a}(\tau) \approx \frac{\kappa}{2} \hat{a}(t). \tag{1.22}$$

With these approximations we can express the equation of motion for the system mode $\hat{a}$ from Eqs. (1.11) and (1.12) as follows,

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{a} &= i\left[\hat{H}_S, \hat{a}\right] - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t}), \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{a} &= i\left[\hat{H}_S, \hat{a}\right] + \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{out}}(\mathrm{t}), \end{aligned}} \begin{aligned} &(1.23) \\ &(1.24) \end{aligned}$$

and get the simplified commutator relations,

$$\left[\hat{b}_{\mathrm{in}}(\mathrm{t}), \hat{b}^{y}_{\mathrm{in}}(t^0)\right] = \left[\hat{b}_{\mathrm{out}}(\mathrm{t}), \hat{b}^{y}_{\mathrm{out}}(t^0)\right] = \delta(t - t^0), \tag{1.25}$$

as well as,

$$\boxed{\hat{b}_{\mathrm{out}}(\mathrm{t}) = \hat{b}_{\mathrm{in}}(\mathrm{t}) + \sqrt{\kappa}\hat{a}(t).} \tag{1.26}$$

The last equality is commonly referred to as the input–output relation. From this expression we can interpret the outgoing mode as the reflection of the input mode together with an additional signal originating from the cavity.

We can further simplify our expressions by assuming that all coupling constants are equal,

$$ig_k := g \in \mathbb{R}, \tag{1.27}$$

and by rewriting the approximated spectral density,

$$\rho(\omega) \approx \rho(\Omega) = g^2 \rho. \tag{1.28}$$

With these simplifications we can write the discretized input-output operators as follows,

$$\hat{b}_{\text{in}}(\text{t}) = \frac{1}{\sqrt{2\pi\rho}} \sum_k e^{-i\omega_k(t-t_0)} \hat{b}_k(t_0), \tag{1.29}$$

$$\hat{b}_{\text{out}}(\text{t}) = \frac{1}{\sqrt{2\pi\rho}} \sum_k e^{i\omega_k(t_N-t)} \hat{b}_k(t_N). \tag{1.30}$$

From these expressions we can intuitively understand the input operator as describing the wave packet of all bath modes from the distant past evolved in time up until the time t. Equivalently, the output operator describes the wave packet consisting of all bath modes in the far future evolved back in time up until the time t. The expectation value $\langle \hat{b}^\dagger_{\text{out}}(t)\, \hat{b}_{\text{out}}(t) \rangle$ for example will give us the number of photons per unit time at the time t in the outgoing wave packet.
Further details on the standard input–output formalism can be found in [4], [3] and [5].

1.3 Keldysh Formalism

In the following we give a short overview of the Keldysh formalism. We will stick closely to the exposition done in [7], recent developments and experimental applications of the formalism can be found in [12]. We follow the notation of [7] and therefore use a bar to denote complex conjugation. We start by introducing the coherent states of the cavity mode $\hat{a}$ and the bath modes $\hat{b}_k$,

$$\hat{a} \left| \phi \right\rangle = \phi \left| \phi \right\rangle, \quad \hat{b}_k \left| \varphi_k \right\rangle = \varphi_k \left| \varphi_k \right\rangle. \tag{1.31}$$

A coherent state in our treatment is defined as follows,

$$\left| \phi \right\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}} \left| n \right\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \left(\hat{a}^\dagger \right)^n \left| 0 \right\rangle = e^{\phi \hat{a}^\dagger} \left| 0 \right\rangle, \tag{1.32}$$

and is not normalized. This leads to the following identity for the inner product of two coherent states,

$$\langle \phi | \phi' \rangle = e^{\bar{\phi}\phi'}. \tag{1.33}$$

Some heavily used identities for coherent states are mentioned in the following,

$$\int d[\phi] e^{-|\phi|^2} \left| \phi \right\rangle \left\langle \phi \right| = \mathbb{1}, \quad d[\phi] := d(\text{Re}\{\phi\}) d(\text{Im}\{\phi\})/\pi, \tag{1.34}$$

$$\left\langle \phi \right| \hat{H}(\hat{a}^\dagger, \hat{a}) \left| \phi \right\rangle = H(\bar{\phi}, \phi') \left\langle \phi | \phi' \right\rangle, \tag{1.35}$$

where the last relation holds for any normally ordered operator $\hat{H}(\hat{a}^\dagger, \hat{a})$. To denote the tensor product of a state from the cavity and the states from the bath we introduce the following notation,

$$\left| \Psi \right\rangle = \left| \phi \right\rangle \bigotimes_k \left| \varphi \right\rangle, \ d[\Psi] = d[\phi] \prod_k d[\varphi_k], \ \bar{\Psi}\Psi' = \bar{\phi}\phi' + \sum_k \bar{\varphi}_k \varphi'_k, \tag{1.36}$$

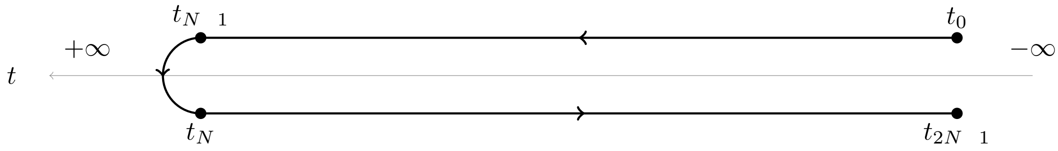

Figure 1.1: The closed time contour of the Keldysh formalism.

with it we express the identity in the combined Hilbert space as follows,

$$\mathbb{1} = \int d[\Psi] e^{-|\Psi|^2} |\Psi\rangle \langle\Psi| . \tag{1.37}$$

As a consequence we compute the trace of an operator $\hat{O}$ as follows,

$$\mathrm{Tr}\Big\{\hat{O}\Big\} = \int d[\Psi] e^{-|\Psi|^2} \langle\Psi| \hat{O} |\Psi\rangle . \tag{1.38}$$

As is customary in quantum field theory, we consider the partition function of our state, here given by the trace of the density matrix $\hat{\rho}$ at time t,

$$Z = \mathrm{Tr}\{\hat{\rho}(t)\} = \mathrm{Tr}\Big\{e^{-i\hat{H}(t-t_0)}\hat{\rho}_0 e^{i\hat{H}(t-t_0)}\Big\} = \mathrm{Tr}\Big\{e^{i\hat{H}(t-t_0)}e^{-i\hat{H}(t-t_0)}\hat{\rho}_0\Big\} = \mathrm{Tr}\{\hat{\rho}_0\} = 1, \tag{1.39}$$

where we used cyclic permutability of the trace and the normalization condition. As a next step, we insert $2N-1$ identities of the type from Eq. (1.37) where we evolve from t_0 to t_{N-1} in the first $N-1$ steps and then backwards in time from $t_{N-1} = t_N$ to $t_{2N-1} = t_0$ in the next $N-1$ steps,

$$\begin{aligned} Z = \int \mathcal{D}[\Psi] e^{-\sum_{j=0}^{2N-1}|\Psi_j|^2} \langle\Psi_{2N-1}| e^{i\delta t\hat{H}} |\Psi_{2N-2}\rangle \langle\Psi_{2N-2}| e^{i\delta t\hat{H}} |\Psi_{2N-3}\rangle \cdots \\ \langle\Psi_{N+1}| e^{i\delta t\hat{H}} |\Psi_N\rangle \langle\Psi_N| \mathbb{1} |\Psi_{N-1}\rangle \langle\Psi_{N-1}| e^{-i\delta t\hat{H}} |\Psi_{N-2}\rangle \cdots \langle\Psi_1| e^{-i\delta t\hat{H}} |\Psi_0\rangle \langle\Psi_0| \hat{\rho}_0 |\Psi_{2N-1}\rangle . \end{aligned} \tag{1.40}$$

See Fig. 1.1 for a sketch of the time contour. We now turn to evaluate the infinitesimal time evolution steps,

$$\begin{aligned} \langle\Psi_j| e^{-i\delta t_j\hat{H}} |\Psi_{j-1}\rangle &= \langle\Psi_j| \Big(\mathbb{1} - i\delta t_j\hat{H} + \mathcal{O}\big(\delta t_j^2\big)\Big) |\Psi_{j-1}\rangle \\ &= e^{\bar{\Psi}_j\Psi_{j-1}} - i\delta t_j H(\bar{\Psi}_j, \Psi_{j-1}) e^{\bar{\Psi}_j\Psi_{j-1}} + \mathcal{O}\big(\delta t_j^2\big) \\ &\approx e^{\bar{\Psi}_j\Psi_{j-1}} e^{-i\delta t_j H(\bar{\Psi}_j,\Psi_{j-1})}. \end{aligned} \tag{1.41}$$

Here we introduced the time–step,

$$\delta t_j = \mathrm{sign}(N-j)\delta t.$$

Notice that the last relation in Eq. (1.41) is exact in the limit $\delta t \to 0$. We further use the common notation for path integrals,

$$\mathcal{D}[\Psi] = \prod_{j=0}^{2N-1} d[\Psi_j]. \tag{1.42}$$

With this notation and the evaluated infinitesimal time step evolution, we can write the Keldysh partition function as a path integral,

$$Z = \int \mathcal{D}[\Psi] e^{iS[\Psi]}, \tag{1.43}$$

$$S[\Psi] = \sum_{j=1}^{2N\ \ 1} \delta t_j \left[i\bar{\Psi}_j \frac{\Psi_j - \Psi_{j\ \ 1}}{\delta t_j} - H(\bar{\Psi}_j, \Psi_{j\ \ 1}) \right] + i\bar{\Psi}_0 \Psi_0 - i \ln \rho_0(\bar{\Psi}_0, \Psi_{2N\ \ 1}). \tag{1.44}$$

1.4 Keldysh Approach to Input–Output Theory

In this section we detail the novel Keldysh path integral approach to input–output theory. Starting from the Keldysh partition function derived in Sec. 1.3, we will derive the Keldysh input–output action for the general scenario introduced in Sec. 1.1 and use it to define generating functionals for the input and output field. From these quantities we will gain access to the statistics of the input and output field.

1.4.1 Keldysh Input-Output Action

We start by inserting the Hamiltonian of our general scenario given in Eq. (1.1) into the expression for the action of the Keldysh partition function from Eq. (1.44) which results in the following action,

$$S[\phi, \varphi] = S_S[\phi] + S_B[\varphi] + S_V[\phi, \varphi]. \tag{1.45}$$

We assume, that the initial state can be separated, $\hat{\rho}_0 = \hat{\rho}_S \otimes \hat{\rho}_B$, and we introduce the tensor product state of all coherent states of the bath at time t_j, $|\varphi_j\rangle = \bigotimes_k |\varphi_{k,j}\rangle$. With this the constituents of the action mentioned above are,

$$S_S[\phi] = \sum_{j=1}^{2N\ \ 1} \delta t_j \left[i\bar{\phi}_j \frac{\phi_j - \phi_{j\ \ 1}}{\delta t_j} - H_S(\bar{\phi}_j, \phi_{j\ \ 1}) \right] + i\bar{\phi}_0 \phi_0 - i \ln \rho_S(\bar{\phi}_0, \phi_{2N\ \ 1}), \tag{1.46}$$

$$\begin{aligned} S_B[\varphi] = &\sum_k \sum_{j=1}^{2N\ \ 1} \delta t_j \left[i\bar{\varphi}_{k,j} \frac{\varphi_{k,j} - \varphi_{k,j\ \ 1}}{\delta t_j} - \omega_k \bar{\varphi}_{k,j} \varphi_{k,j\ \ 1} \right] \\ &+ i \sum_k \varphi_{k,0} \bar{\varphi}_{k,0} - i \ln \rho_B(\bar{\varphi}_0, \varphi_{2N\ \ 1}), \end{aligned} \tag{1.47}$$

$$S_V[\phi, \varphi] = -\sum_k \sum_{j=1}^{2N\ \ 1} \delta t_j \left[g_k \bar{\phi}_j \varphi_{k,j\ \ 1} + g_k \bar{\varphi}_{k,j} \phi_{j\ \ 1} \right]. \tag{1.48}$$

Here we used the following notation,

$$\rho(a, b) = \langle a| \hat{\rho} |b\rangle .$$

In the following steps we will often make use of the bosonic Gaussian integral, the formula for which can be found in App. A.2.1.

Since in input–output theory we are only interested in the bath states in the distant past and future, we only need the information of the bath modes at $t_0 = t_{2N\ \ 1}$, which lie in the distant past, and $t_N = t_{N\ \ 1}$, which lie in the distant future. Therefore we perform the path integral over the bath modes $\varphi_{k,1},\dots,\varphi_{k,N-2},\varphi_{k,N+1},\dots,\varphi_{k,2N-2}$ for all values of k. The detailed calculation of this integral can be found in the appendix, see App. A.2.2. We further introduce the following notation to differentiate between fields on the forward and backwards part of the time contour,

$$\phi_j^+ = \phi_j, \qquad \phi_j^- = \phi_{2N-1-j}, \qquad \text{for } j \in \{0,\dots,N-1\}, \tag{1.49}$$

and, analogously to the input and output operators in Eq. (1.8), we define the input and output fields,

$$\varphi_{\mathrm{in},j}^+ = \frac{1}{\sqrt{\kappa}}\sum_k i g_k e^{-i\omega_k(t_j-t_0)}\varphi_{k,0}, \qquad \varphi_{\mathrm{in},j}^- = \frac{1}{\sqrt{\kappa}}\sum_k i g_k e^{-i\omega_k(t_j-t_0)}\varphi_{k,2N-1}, \tag{1.50}$$

$$\varphi_{\mathrm{out},j}^+ = \frac{1}{\sqrt{\kappa}}\sum_k i g_k e^{i\omega_k(t_N-t_j)}\varphi_{k,N-1}, \qquad \varphi_{\mathrm{out},j}^- = \frac{1}{\sqrt{\kappa}}\sum_k i g_k e^{i\omega_k(t_N-t_j)}\varphi_{k,N}. \tag{1.51}$$

After performing the integration over the mentioned bath modes we get the following expression for the partition function,

$$Z = \int \left(\prod_{j=0}^{2N-1} d[\phi_j]\right)\left(\prod_k d[\varphi_{k,0}]d[\varphi_{k,N-1}]d[\varphi_{k,N}]d[\varphi_{k,2N-1}]\right) e^{i\tilde{S}[\phi,\varphi]}, \tag{1.52}$$

with

$$\tilde{S}[\phi,\varphi] = \tilde{S}_S[\phi] + \tilde{S}_B[\varphi] + \tilde{S}_V[\phi,\varphi], \tag{1.53}$$

$$\tilde{S}_S[\phi] = S_S[\phi] + i\delta t^2 \sum_{j=1}^{N-2}\sum_{l=1}^{j}\int_0^{\infty} d\omega\rho(\omega)\left[\bar{\phi}_{j+1}^+\phi_{l-1}^+ e^{-i\omega(t_j-t_l)} + \bar{\phi}_{l-1}^-\phi_{j+1}^- e^{i\omega(t_j-t_l)}\right], \tag{1.54}$$

$$\begin{aligned}\tilde{S}_B[\varphi] = {} & i\sum_k \begin{pmatrix}\bar{\varphi}_{k,0}\\ \bar{\varphi}_{k,N-1}\\ \bar{\varphi}_{k,N}\\ \bar{\varphi}_{k,2N-1}\end{pmatrix}^T \begin{pmatrix}1 & 0 & 0 & 0\\ -e^{-i\omega_k(t_N-t_0)} & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -e^{i\omega_k(t_N-t_0)} & 1\end{pmatrix} \begin{pmatrix}\varphi_{k,0}\\ \varphi_{k,N-1}\\ \varphi_{k,N}\\ \varphi_{k,2N-1}\end{pmatrix}\\ & - i\ln\rho_B(\bar{\varphi}_0,\varphi_{2N-1}),\end{aligned} \tag{1.55}$$

$$S_V[\phi,\varphi] = i\delta t\sqrt{\kappa}\sum_{j=1}^{N-1}\left[\bar{\phi}_j^+\varphi_{\mathrm{in},j-1}^+ - \bar{\varphi}_{\mathrm{out},j}^+\phi_{j-1}^+ - \bar{\phi}_{j-1}^-\varphi_{\mathrm{out},j}^- + \bar{\varphi}_{\mathrm{in},j-1}^-\phi_j^-\right]. \tag{1.56}$$

Where we employed the spectral density from Eq. (1.9) in the second equation. Analogously to the procedure in the discretized version of standard input–output theory from Sec. 1.2 we can assume the bath modes to be equidistantly spaced such that

$$\omega_k = \omega_0 + k\delta\omega, \qquad \delta\omega\delta t = \frac{2\pi}{N}, \qquad k \in \{0,\dots,N-1\}, \tag{1.57}$$

and invert the input–output fields to retrieve the bath modes at the initial or final time,

$$\varphi_{k,0} = \frac{\sqrt{\kappa}}{i g_k}\frac{1}{N}\sum_{j=0}^{N-1} e^{i\omega_k(t_j-t_0)}\varphi_{\mathrm{in},j}^+, \qquad \varphi_{k,2N-1} = \frac{\sqrt{\kappa}}{i g_k}\frac{1}{N}\sum_{j=0}^{N-1} e^{i\omega_k(t_j-t_0)}\varphi_{\mathrm{in},j}^-, \tag{1.58}$$

$$\varphi_{k,N-1} = \frac{\sqrt{\kappa}}{i g_k}\frac{1}{N}\sum_{j=0}^{N-1} e^{-i\omega_k(t_N-t_j)}\varphi_{\mathrm{out},j}^+, \qquad \varphi_{k,N} = \frac{\sqrt{\kappa}}{i g_k}\frac{1}{N}\sum_{j=0}^{N-1} e^{-i\omega_k(t_N-t_j)}\varphi_{\mathrm{out},j}^-. \tag{1.59}$$

These relations can now be used to express the action $S[\phi,\varphi]$ completely in terms of the input-output fields. Additional to the assumption of equidistant spacing of the bath modes we now assume a flat spectral density,

$$ig_k = g \in \mathbb{R}\ \forall\, k, \tag{1.60}$$

and express the coupling strength κ through the spectral density,

$$\kappa = \frac{2\pi g^2}{\delta\omega} = g^2 N\delta t. \tag{1.61}$$

Now we can for instance rewrite,

$$\begin{aligned}\sum_{k=0}^{N-1}\bar\varphi_{k,0}\varphi_{k,0} &= \sum_{k=0}^{N-1}\frac{\kappa}{|g_k|^2}\frac{1}{N^2}\sum_{j,l=0}^{N-1} e^{-i\omega_k(t_j-t_l)}\bar\varphi^+_{\mathrm{int},j}\bar\varphi^+_{\mathrm{in},l} = \sum_{k=0}^{N-1}\frac{\delta t}{N}\sum_{j,l=0}^{2N-1} e^{-i\omega_k(t_j-t_l)}\bar\varphi^+_{\mathrm{in},j}\varphi^+_{\mathrm{in},l}\\ &= \sum_{k=0}^{N-1}\frac{\delta\omega\delta t^2}{2\pi}\sum_{j,l=0}^{2N-1} e^{-i\omega_k(t_j-t_l)}\bar\varphi^+_{\mathrm{in},j}\varphi^+_{\mathrm{in},l} = \sum_{j=0}^{N-1}\delta t\bar\varphi^+_{\mathrm{in},j}\varphi^+_{\mathrm{in},j},\end{aligned} \tag{1.62}$$

where in the last two steps we used the equidistant spacing from Eq. (1.57) and the properties of the Dirac delta distribution (for more detail see (A.1) in the appendix), which yields a factor of $2\pi\frac{\delta_{j,l}}{\delta t}$ in the limit of continuous time. We now turn to the spectral density in $S_S[\phi]$. Analogously to the treatment in Sec. 1.2 we make the assumption that $\rho(\omega)\approx\rho(\Omega)=2\pi\kappa$ which results in,

$$\int_0^{1} d\omega\rho(\omega)e^{-i\omega(t_j-t_l)} = \kappa\frac{\delta_{j,l}}{\delta t}. \tag{1.63}$$

This again leads us to make the replacement,

$$\sum_{j=0}^{N-1}\sum_{j=0}^{l}\delta t^2 f_{j,l}\int_0^{1} d\omega\rho(\omega)e^{-i\omega(t_j-t_l)} \to \frac{1}{2}\sum_{j=0}^{N-1}\delta t\kappa f_{j,j}, \tag{1.64}$$

where $f_{j,l}$ is an arbitrary function of the two indices and where we used the discrete version of the identity given in Eq. (A.2). With these simplifications we find the Keldysh input-output action,

$$S^{\mathrm{io}}[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}] = S^{\mathrm{io}}_S[\phi] + S^{\mathrm{io}}_B[\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}] + S^{\mathrm{io}}_V[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}], \tag{1.65}$$

where the constituents are,

$$S^{\mathrm{io}}_S[\phi] = S_S[\phi] + i\frac{\kappa}{2}\sum_{j=1}^{N-2}\delta t\left[\bar\phi^+_{j+1}\phi^+_{j-1} + \bar\phi_{j-1}\phi_{j+1}\right], \tag{1.66}$$

$$\begin{aligned}S^{\mathrm{io}}_B[\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}] &= i\sum_{j=0}^{N-1}\delta t\begin{pmatrix}\bar\varphi^+_{\mathrm{in},j} & \bar\varphi^+_{\mathrm{out},j} & \bar\varphi_{\mathrm{out},j} & \bar\varphi_{\mathrm{in},j}\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}\varphi^+_{\mathrm{in},j}\\ \varphi^+_{\mathrm{out},j}\\ \varphi_{\mathrm{out},j}\\ \varphi_{\mathrm{in},j}\end{pmatrix} \\ &\quad - i\ln\rho_B(\varphi_{\mathrm{in}}),\end{aligned} \tag{1.67}$$

$$\begin{aligned}S^{\mathrm{io}}_V[\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}] &= S_V[\phi,\tilde\varphi]\\ &= i\delta t\sqrt{\kappa}\sum_{j=1}^{N-1}\left[\bar\phi^+_j\varphi^+_{\mathrm{in},j-1} - \bar\varphi^+_{\mathrm{out},j}\phi^+_{j-1} - \bar\phi_{j-1}\varphi_{\mathrm{out},j} + \bar\varphi_{\mathrm{in},j-1}\phi_j\right].\end{aligned} \tag{1.68}$$

In the continuous limit these expression are,

$$S_S^{\text{io}}[\phi] = \int_{t_0}^{t_N} dt\,\Big[\bar{\phi}^+(t)\Big(i\partial_t + i\frac{\kappa}{2}\Big)\phi^+(t) - \bar{\phi}\;\;(t)\Big(i\partial_t - i\frac{\kappa}{2}\Big)\phi\;\;(t) \tag{1.69}$$

$$- H_S(\bar{\phi}^+(t), \phi^+(t)) + H_S(\bar{\phi}\;\;(t), \phi\;\;(t))\Big]\,,$$

$$S_B^{\text{io}}[\varphi_{\text{in}}, \varphi_{\text{out}}] = i\int_{t_0}^{t_N} dt\big(\bar{\varphi}_{\text{in}}^+(t)\;\bar{\varphi}_{\text{out}}^+(t)\;\bar{\varphi}_{\text{out}}(t)\;\bar{\varphi}_{\text{in}}(t)\big)\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_{\text{in}}^+(t) \\ \varphi_{\text{out}}^+(t) \\ \varphi_{\text{out}}(t) \\ \varphi_{\text{in}}(t) \end{pmatrix} \tag{1.70}$$

$$- i\ln\rho_B(\varphi_{\text{in}}),$$

$$S_V^{\text{io}}[\varphi_{\text{in}}, \varphi_{\text{out}}] = i\sqrt{\kappa}\int_{t_0}^{t_N} dt\big[\bar{\phi}^+(t)\varphi_{\text{in}}^+(t) - \bar{\varphi}_{\text{out}}^+(t)\phi^+(t) - \bar{\phi}\;\;(t)\varphi_{\text{out}}(t) + \bar{\varphi}_{\text{in}}(t)\phi\;\;(t)\big]. \tag{1.71}$$

Here we dropped the boundary term in $S_S^{\text{io}}[\phi]$ since its influence should be negligible in the presence of dissipation and under the assumption that $t_0 \to -\infty$. The differential element is now given as follows,

$$\mathcal{D}[\phi, \varphi_{\text{in}}, \varphi_{\text{out}}] = \mathcal{D}[\phi]\mathcal{D}[\varphi_{\text{in}}]\mathcal{D}[\varphi_{\text{out}}]$$

$$= \left(\prod_{j=0}^{N\;\;1} d[\phi_j^+]d[\phi_j\;\;]\right)\left(\prod_{j=0}^{N\;\;1} d[\varphi_{\text{in},j}^+]d[\varphi_{\text{in},j}]\right)\left(\prod_{j=0}^{N\;\;1} d[\varphi_{\text{out},j}^+]d[\varphi_{\text{out},j}]\right). \tag{1.72}$$

1.4.2 Input States

In this subsection we will consider different input states. The input state here is the initial state of the bath $\hat{\rho}_B(\varphi_{\text{in}})$ that appears in the bath part of the input–output action in Eq. (1.70). The thermal state will play a central role in the following, for a definition and important properties see A.2.3.

Thermal State

We firstly consider the case where the bath is in a thermal state at t_0,

$$\rho_B(\bar{\varphi}_0, \varphi_{2N\;\;1}) = \exp\left[\sum_k e^{\;\;\beta\omega_k}\bar{\varphi}_{k,0}\varphi_{k,2N\;\;1}\right]\prod_k\big(1 - e^{\;\;\beta\omega_k}\big). \tag{1.73}$$

We further assume that all frequencies interacting with the bath can be set constant $\omega_k \approx \Omega\ \forall k$ and use the relation introduced in Eq. (1.62) to express the initial state through the input field,

$$\rho_B(\varphi_{\text{in}}) = \exp\left[e^{\;\;\beta\Omega}\sum_{j=0}^{N\;\;1}\delta t\bar{\varphi}_{\text{in},j}^+\varphi_{\text{in},j}\right]\big(1 - e^{\;\;\beta\Omega}\big)^N. \tag{1.74}$$

Product of Coherent States

For the input being a product of normalized coherent states,

$$\hat{\rho}_B = \bigotimes_k e^{\;\;j\alpha_k j^2}\,|\alpha_k\rangle\,\langle\alpha_k|\,, \tag{1.75}$$

we find,

$$
\begin{aligned}
\rho_B(\bar{\varphi}_0, \varphi_{2N\ \ 1}) &= \left(\bigotimes_k \langle \varphi_{k,0}|\right)\left(\bigotimes_k e^{\ \ j\alpha_k j^2} |\alpha_k\rangle \langle \alpha_k|\right)\left(\bigotimes_k |\varphi_{k,2N\ \ 1}\rangle\right) \\
&= \prod_k e^{\ \ j\alpha_k j^2} e^{\bar{\varphi}_{k,0}} e^{\bar{\alpha}_k \varphi_{k,2N\ \ 1}} = \exp\left[\sum_k \bar{\varphi}_{k,0}\alpha_k + \bar{\alpha}_k \varphi_{k,2N\ \ 1} - |\alpha_k|^2\right] \\
&= \exp\left[\sum_{j=0}^{N\ \ 1}\left(\bar{\varphi}^+_{\mathrm{in},j} \sum_k \frac{\sqrt{\delta t}}{\sqrt{N}} e^{\ \ i\omega_k(t_j\ \ t_0)}\alpha_k\right.\right. \\
&\qquad \left.\left. + \varphi_{\mathrm{in},j} \sum_k \frac{\sqrt{\delta t}}{\sqrt{N}} e^{i\omega_k(t_j\ \ t_0)}\bar{\alpha}_k - \sum_k |\alpha_k|^2\right)\right]. \qquad (1.76)
\end{aligned}
$$

We introduce the function f to encode the information over the input state,

$$
f_j = \sum_{k=0}^{N\ \ 1} \frac{\alpha_k}{\sqrt{N\delta t}} e^{\ \ i\omega_k(t_j\ \ t_0)}, \qquad (1.77)
$$

with this we can express the input state as follows,

$$
\rho_B(\varphi_{\mathrm{in}}) = \exp\left[\sum_{j=0}^{N\ \ 1} \delta t \Big(f_j \bar{\varphi}^+_{\mathrm{in},j} + \bar{f}_j \varphi_{\mathrm{in},j} - |f_j|^2\Big)\right]. \qquad (1.78)
$$

Details on this step can be found in A.2.3.

Through the coherent states α_k, we can now implement an arbitrary function f_j in the input state. We mention two important examples in the following,
<u>Pulse of vanishing temporal width:</u>

$$
\alpha_k = \alpha\sqrt{\frac{\delta t}{N}} e^{i\omega_k(t_l\ \ t_0)} \Rightarrow \qquad f_j = \delta_{j,l}\alpha. \qquad (1.79)
$$

<u>Monochromatic coherent signal:</u>

$$
\alpha_k = \alpha\sqrt{N\delta t}\delta_{k,q} \qquad \Rightarrow \qquad f_j = \alpha e^{\ \ i\omega_q(t_j\ \ t_0)}. \qquad (1.80)
$$

Displaced Thermal State

The density matrix of a displaced thermal state is given as follows,

$$
\hat{\rho}_B = \bigotimes_k \left(1 - e^{\ \ \beta\omega_k}\right) \hat{D}(\alpha_k) e^{\ \ \beta\omega_k \hat{a}_k^{y} \hat{a}_k} \hat{D}^{y}(\alpha_k). \qquad (1.81)
$$

Where we use the displacement operator $\hat{D}$, a definition and important properties can be found in the appendix, see A.2.3. We again evaluate the contribution in the input–output action,

$$
\begin{aligned}
\rho_B(\varphi_0, \varphi_{2N\ \ 1}) &= \langle \varphi_0| \hat{\rho}_B |\varphi_{2N\ \ 1}\rangle \\
&= \bigotimes_k \left(1 - e^{\ \ \beta\omega_k}\right) \langle \varphi_{0,k}| \hat{D}(\alpha_k) e^{\ \ \beta\omega_k \hat{a}_k^{y} \hat{a}_k} \hat{D}^{y}(\alpha_k) |\varphi_{2N\ \ 1,k}\rangle \\
&= \bigotimes_k \left(1 - e^{\ \ \beta\omega_k}\right)\Delta_k \qquad (1.82)
\end{aligned}
$$

using the relation from Eq. (A.41) above we get,

$$\Delta_k = e^{\ \ j\alpha_k j^2 + \varphi_{k,2N\ \ 1}\bar{\alpha}_k + \bar{\varphi}_{k,0}\alpha_k} \langle \varphi_{k,0} - \alpha_k | \, e^{\ \ \beta\omega_k \hat{a}_k^y \hat{a}_k} \, | \varphi_{k,2N\ \ 1} - \alpha_k \rangle \, . \tag{1.83}$$

Further using the relation from Eq. (A.28) we find,

$$\begin{aligned}\rho_B(\varphi_0, \varphi_{2N\ \ 1}) = &\left[\prod_k e^{\ \ j\alpha_k j^2 (1\ \ e^{\beta\omega_k})} (1 - e^{\ \ \beta\omega_k}) \right] \\ &\exp\left[\sum_k \left[e^{\ \ \beta\omega_k} \bar{\varphi}_{k,0}\varphi_{k,2N\ \ 1} + \left(1 - e^{\ \ \beta\omega_k}\right)\alpha_k \bar{\varphi}_{k,0} + \left(1 - e^{\ \ \beta\omega_k}\right)\bar{\alpha}_k \varphi_{k,2N\ \ 1} \right] \right] . \end{aligned} \tag{1.84}$$

Again we assume a flat spectral density for the relevant modes, $\exp(-\beta\omega_k) \simeq \exp(-\beta\Omega)$, which in the continuum limit leads to,

$$\begin{aligned}\rho_B(\varphi_0, \varphi_{2N\ \ 1}) = &\left(1 - e^{\ \ \beta\Omega}\right)^N \\ &\exp\left[\int_{t_0}^{t_N} dt \left[e^{\ \ \beta\Omega} \bar{\varphi}_{\text{in}}^+(t)\varphi_{\text{in}}(t) + \left(1 - e^{\ \ \beta\Omega}\right)\left(f(t)\bar{\varphi}_{\text{in}}^+(t) + \bar{f}(t)\varphi_{\text{in}}(t) - |f(t)|^2 \right) \right] \right] . \end{aligned} \tag{1.85}$$

Note that N goes to infinity in the continuum limit and the prefactor $\left(1 - e^{\ \ \beta\Omega}\right)^N$ is simply a shorthand for the normalization constant. From this expression we can also arrive at the continuum limit of the thermal state from Eq. (1.74) by setting f to zero and at the limit of the displaced coherent state by setting the temperature to zero.

1.4.3 Reduced Action for the System

In this subsection we consider the reduced action for our system, i.e. we integrate out the input and output modes to achieve a description of the system alone that we can check against existing descriptions of our setting. We begin by integrating out the output modes from the input-output action S^{io}. S_S^{io} contains no input fields so we focus on S_B^{io} and S_V^{io} and find in the exponent,

$$\begin{aligned} S^{\text{i}}[\phi, \varphi_{\text{in}}] = &S_S^{\text{io}} + i\delta t \sum_{j=0}^{N\ \ 1} \left[\bar{\varphi}_{\text{in},j}^+ \varphi_{\text{in},j}^+ + \bar{\varphi}_{\text{in},j}\varphi_{\text{in},j} + \sqrt{\kappa}\left(\bar{\phi}_j^+ \varphi_{\text{in},j\ \ 1}^+ + \bar{\varphi}_{\text{in},j\ \ 1}\phi_j \right) \right] \\ &- i\delta t \sum_{j=1}^{N\ \ 1} \left[\bar{\varphi}_{\text{in},j}\varphi_{\text{in},j}^+ + \sqrt{\kappa}\left(\bar{\varphi}_{\text{in},j}\phi_{j\ \ 1}^+ + \bar{\phi}_{j\ \ 1}\varphi_{\text{in},j}^+ \right) + \kappa\bar{\phi}_{j\ \ 1}\phi_{j\ \ 1}^+ \right] - i \ln \rho_B(\varphi_{\text{in}}). \end{aligned} \tag{1.86}$$

We introduce a new pair of fields,

$$\varphi^{\text{cl}}(t) = \frac{1}{\sqrt{2}}\left(\varphi^+(t) + \varphi\ \ (t)\right), \qquad \varphi^{\text{q}}(t) = \frac{1}{\sqrt{2}}\left(\varphi^+(t) - \varphi\ \ (t)\right). \tag{1.87}$$

The superscripts 'cl' and 'q' stand for the classical and quantum components of the field. Introducing these fields is often referred to as a Keldysh rotation [7].
Calculating the necessary integrals and performing a Keldysh rotation as well as going to the

continuum limit we find,

$$S^{\mathrm{i}}[\phi,\varphi_{\mathrm{in}}] = S^{\mathrm{i}}_S[\phi] + S^{\mathrm{i}}_B[\varphi_{\mathrm{in}}] + S^{\mathrm{i}}_V[\phi,\varphi_{\mathrm{in}}], \tag{1.88}$$

$$S^{\mathrm{i}}_S[\phi] = S_S[\phi] + i\kappa\int_{t_0}^{t_N} dt\Big[\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{q}}(t) - \frac{1}{2}\bar{\phi}^{\mathrm{cl}}(t)\phi^{\mathrm{q}}(t) + \frac{1}{2}\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{cl}}(t)\Big], \tag{1.89}$$

$$S^{\mathrm{i}}_B[\varphi_{\mathrm{in}}] = i\int_{t_0}^{t_N} dt\big[\bar{\varphi}^{+}_{\mathrm{in}}(t)\varphi^{+}_{\mathrm{in}}(t) + \bar{\varphi}_{\mathrm{in}}(t)\varphi_{\mathrm{in}}(t) - \bar{\varphi}_{\mathrm{in}}(t)\varphi^{+}_{\mathrm{in}}(t)\big] - i\ln\rho_B(\varphi_{\mathrm{in}}), \tag{1.90}$$

$$S^{\mathrm{i}}_V[\phi,\varphi_{\mathrm{in}}] = i\sqrt{2\kappa}\int_{t_0}^{t_N} dt\big[\bar{\phi}^{\mathrm{q}}(t)\varphi^{+}_{\mathrm{in}}(t) - \bar{\varphi}_{\mathrm{in}}(t)\phi^{\mathrm{q}}(t)\big]. \tag{1.91}$$

To integrate out the input modes as well, we revert back to the discrete notation and evaluate the contribution from the boundary term,

$$\ln\Big(\rho_B(\varphi_{\mathrm{in}})\big(1-e^{\;\beta\Omega}\big)^{\;N}\Big) = \\ \sum_{j=0}^{N\;1} e^{\;\beta\Omega}\delta t\bar{\varphi}^{+}_{\mathrm{in},j}\varphi_{\mathrm{in},j} + \big(1-e^{\;\beta\Omega}\big)\delta t\Big[f_j\bar{\varphi}^{+}_{\mathrm{in},j} + \bar{f}_j\varphi_{\mathrm{in},j} - |f_j|^2\Big]. \tag{1.92}$$

The factor $\big(1-e^{\;\beta\Omega}\big)^{\;N}$ serves as a normalization and will drop out of the integral during the integration. We will neglect it from now on. This leads us to,

$$\begin{aligned} i\big(S^{\mathrm{i}}_B[\varphi_{\mathrm{in}}] + S^{\mathrm{i}}_V[\phi,\varphi_{\mathrm{in}}]\big) = &-\delta t\sum_{j=0}^{N\;1}\big(\bar{\varphi}^{+}_{\mathrm{in},j}\;\bar{\varphi}_{\mathrm{in},j}\big)\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\varphi^{+}_{\mathrm{in},j}\\ \varphi_{\mathrm{in},j}\end{pmatrix} \\ &+\sum_{j=0}^{N\;1} e^{\;\beta\Omega}\delta t\bar{\varphi}^{+}_{\mathrm{in},j}\varphi_{\mathrm{in},j} + \big(1-e^{\;\beta\Omega}\big)\delta t\Big[f_j\bar{\varphi}^{+}_{\mathrm{in},j} + \bar{f}_j\varphi_{\mathrm{in},j} - |f_j|^2\Big] \\ &-\delta t\sum_{j=1}^{N}\sqrt{\kappa}\Big(\bar{\phi}^{+}_j\varphi^{+}_{\mathrm{in},j\;\;1} + \bar{\varphi}_{\mathrm{in},j\;\;1}\phi_j - \bar{\varphi}_{\mathrm{in},j\;\;1}\phi^{+}_{j\;\;1} - \bar{\phi}_{j\;\;1}\varphi^{+}_{\mathrm{in},j}\Big). \end{aligned} \tag{1.93}$$

As usual we employ the formula for the Gaussian integral, and performing the continuum limit in all terms, we get the reduced action,

$$\begin{aligned} S[\phi] =& S_S[\phi] + i\sqrt{2\kappa}\int_{t_0}^{t_N} dt\big(f(t)\bar{\phi}^{\mathrm{q}}(t) - \bar{f}(t)\phi^{\mathrm{q}}(t)\big) \\ &+ i\kappa\int_{t_0}^{t_N} dt\Big[(2n_B+1)\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{q}}(t) - \frac{1}{2}\bar{\phi}^{\mathrm{cl}}(t)\phi^{\mathrm{q}}(t) + \frac{1}{2}\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{cl}}(t)\Big], \end{aligned} \tag{1.94}$$

where we introduced the occupation number of the bath,

$$n_B = \frac{1}{e^{\beta\Omega}-1}. \tag{1.95}$$

The second term in Eq. (1.94) corresponds to a coherent Hamiltonian drive term of the following form,

$$\hat{H}_D(t) = -i\sqrt{\kappa}\big(f(t)\hat{a}^{y} - \bar{f}(t)\hat{a}\big), \tag{1.96}$$

and the last term corresponds to the action of a Markovian thermal bath [13]. The discrete version of the action in Eq. (1.94) above is

$$
\begin{aligned}
S[\phi] =& S_S[\phi] + i\frac{\kappa}{2}\sum_{j=1}^{N\ \ 1} \delta t \big[\bar{\phi}^{+}_{j+1}\phi^{+}_{j\ \ 1} + \bar{\phi}_{j\ \ 1}\phi_{j+1} - 2\bar{\phi}_{j\ \ 1}\phi^{+}_{j+1}\big] \\
&+ \delta t\sqrt{\kappa}\sum_{j=1}^{N\ \ 2} \big[f_j\big(\bar{\phi}_{j\ \ 1} - \bar{\phi}^{+}_{j+1}\big) + \bar{f}_j\big(\phi^{+}_{j\ \ 1} + \phi_{j+1}\big)\big] \\
&+ \delta t\kappa n_B \sum_{j=1}^{N\ \ 2} \big(\bar{\phi}_{j\ \ 1} - \bar{\phi}^{+}_{j+1}\big)\big(\phi^{+}_{j\ \ 1} - \phi_{j+1}\big),
\end{aligned}
\tag{1.97}
$$

with

$$
\begin{aligned}
S_S[\phi] = &\sum_{j=1}^{N\ \ 1}\delta t\left[i\bar{\phi}^{+}_{j}\frac{\phi^{+}_{j} - \phi^{+}_{j\ \ 1}}{\delta t} - H_S(\bar{\phi}^{+}_{j}, \phi^{+}_{j\ \ 1})\right] + \sum_{j=1}^{N\ \ 1}\delta t\left[i\bar{\phi}_{j}\ \frac{\phi_{j}\ - \phi_{j\ \ 1}}{\delta t} + H_S(\bar{\phi}_{j}\ , \phi_{j\ \ 1})\right] \\
&+ i\bar{\phi}_0\phi_0 - i\ln\rho_S(\bar{\phi}_0, \phi_{2N\ \ 1}).
\end{aligned}
\tag{1.98}
$$

1.4.4 Stationary Phase Approximation

In this subsection we derive a relation between the system field and the input and output fields in the semi-classical limit of our field theoretic approach. In the semiclassical limit where $\hbar$ tends to zero, our system is expected to take those trajectories through its state space where the action is stationary, hence this is called the stationary phase approximation. The relations we derive with this approach will be the analogue to the input–output relation from the standard input–output theory from Sec. 1.2. With the Keldysh rotation, we can rewrite the input-output action as follows,

$$
S^{\mathrm{io}}_{B}[\varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}] = i\int_{t_0}^{t_N} dt \begin{pmatrix} \bar{\varphi}^{\mathrm{cl}}_{\mathrm{in}}(t) \\ \bar{\varphi}^{\mathrm{q}}_{\mathrm{in}}(t) \\ \bar{\varphi}^{\mathrm{cl}}_{\mathrm{out}}(t) \\ \bar{\varphi}^{\mathrm{q}}_{\mathrm{out}}(t) \end{pmatrix}^{T} \begin{pmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 3/2 \end{pmatrix} \begin{pmatrix} \varphi^{\mathrm{cl}}_{\mathrm{in}}(t) \\ \varphi^{\mathrm{q}}_{\mathrm{in}}(t) \\ \varphi^{\mathrm{cl}}_{\mathrm{out}}(t) \\ \varphi^{\mathrm{q}}_{\mathrm{out}}(t) \end{pmatrix} \tag{1.99}
$$

$$
- i\ln\rho_B(\varphi_{\mathrm{in}}),
$$

$$
\begin{aligned}
S^{\mathrm{io}}_{V}[\phi, \varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}] = i\frac{\sqrt{\kappa}}{2}\int_{t_0}^{t_N} dt\ &\big[\big(\bar{\phi}^{\mathrm{cl}}(t) + \bar{\phi}^{\mathrm{q}}(t)\big)\big(\varphi^{\mathrm{cl}}_{\mathrm{in}}(t) + \varphi^{\mathrm{q}}_{\mathrm{in}}(t)\big) \\
&- \big(\bar{\varphi}^{\mathrm{cl}}_{\mathrm{out}}(t) + \bar{\varphi}^{\mathrm{q}}_{\mathrm{out}}(t)\big)\big(\phi^{\mathrm{cl}}(t) + \phi^{\mathrm{q}}(t)\big) \\
&- \big(\bar{\phi}^{\mathrm{cl}}(t) - \bar{\phi}^{\mathrm{q}}(t)\big)\big(\varphi^{\mathrm{cl}}_{\mathrm{out}}(t) - \varphi^{\mathrm{q}}_{\mathrm{out}}(t)\big) \\
&+ \big(\bar{\varphi}^{\mathrm{cl}}_{\mathrm{in}}(t) - \bar{\varphi}^{\mathrm{q}}_{\mathrm{in}}(t)\big)\big(\phi^{\mathrm{cl}}(t) - \phi^{\mathrm{q}}(t)\big)\big].
\end{aligned}
\tag{1.100}
$$

We are now interested in the relation between the input and output fields with the system fields and each other for a stationary phase. To determine this, we compute the functional derivatives of the input–output action, that we will simply refeer to as the action S in the following. For a definition of the functional derivative, see A.2.4.

$$i\frac{\delta S}{\delta\varphi^{\text{cl}}_{\text{out}}(t)} = -\frac{1}{2}\bar{\varphi}^{\text{cl}}_{\text{out}}(t) + \frac{1}{2}\bar{\varphi}^{\text{cl}}_{\text{in}}(t) - \frac{1}{2}\bar{\varphi}^{\text{q}}_{\text{in}}(t) - \frac{1}{2}\bar{\varphi}^{\text{q}}_{\text{out}}(t) + \frac{\sqrt{\kappa}}{2}\big(\bar{\phi}^{\text{cl}}(t) - \bar{\phi}^{\text{q}}(t)\big), \tag{1.101}$$

$$i\frac{\delta S}{\delta\varphi^{\text{q}}_{\text{out}}(t)} = -\frac{3}{2}\bar{\varphi}^{\text{q}}_{\text{out}}(t) + \frac{1}{2}\bar{\varphi}^{\text{cl}}_{\text{out}}(t) - \frac{1}{2}\bar{\varphi}^{\text{cl}}_{\text{in}}(t) + \frac{1}{2}\bar{\varphi}^{\text{q}}_{\text{in}}(t) - \frac{\sqrt{\kappa}}{2}\big(\bar{\phi}^{\text{cl}}(t) - \bar{\phi}^{\text{q}}(t)\big), \tag{1.102}$$

$$i\frac{\delta S}{\delta\bar{\varphi}^{\text{cl}}_{\text{out}}(t)} = -\frac{1}{2}\varphi^{\text{cl}}_{\text{out}}(t) + \frac{1}{2}\varphi^{\text{q}}_{\text{in}}(t) + \frac{1}{2}\varphi^{\text{q}}_{\text{in}}(t) + \frac{1}{2}\varphi^{\text{q}}_{\text{out}}(t) + \frac{\sqrt{\kappa}}{2}\big(\phi^{\text{cl}}(t) + \phi^{\text{q}}(t)\big), \tag{1.103}$$

$$i\frac{\delta S}{\delta\bar{\varphi}^{\text{q}}_{\text{out}}(t)} = -\frac{3}{2}\varphi^{\text{q}}_{\text{out}}(t) - \frac{1}{2}\varphi^{\text{cl}}_{\text{out}}(t) + \frac{1}{2}\varphi^{\text{cl}}_{\text{in}}(t) + \frac{1}{2}\varphi^{\text{q}}_{\text{in}}(t) + \frac{\sqrt{\kappa}}{2}\big(\phi^{\text{cl}}(t) + \phi^{\text{q}}(t)\big). \tag{1.104}$$

Setting these equations to zero and combining them we find,

$$\boxed{\varphi^{\text{cl}}_{\text{out}}(t) = \varphi^{\text{cl}}_{\text{in}}(t) + \sqrt{\kappa}\phi^{\text{cl}}(t),} \tag{1.105}$$

$$\boxed{\varphi^{\text{q}}_{\text{out}}(t) = \varphi^{\text{q}}_{\text{in}}(t) + \sqrt{\kappa}\phi^{\text{q}}(t) = 0.} \tag{1.106}$$

These equations are the analogue of the input-output relations from the standard input-output theory,

$$\boxed{\hat{b}_{\text{out}}(\text{t}) = \hat{b}_{\text{in}}(\text{t}) + \sqrt{\kappa}\hat{a}(t).} \tag{1.107}$$

The fact that the quantum part of the field is zero here is a consequence of the stationary phase approximation. A stationary phase means that we take the classical trajectory and no coherences are built up.

1.4.5 Statistics of the Input Field

We now turn to deriving the statistics of the input field from our generating functional Z. This in itself will not lead to information about the system but will serve as a way of introducing the methods with which we will gain access to the statistics of the output field later on. After integrating out φ_{out} and ϕ the action reduces to the part dependend only on the input bath modes defined in Eq. (1.90), $S^{\text{i}}_{B}[\varphi_{\text{in}}]$. This can heuristically be derived as follows; the path integral is always normalized to one, even for a non-Hermitian Hamiltonian, and $S^{\text{i}}_{V}[\phi, \varphi_{\text{in}}]$ looks like the system action from a non-Hermitian Hamiltonian,

$$\hat{H} = -i\sqrt{\kappa}\big(\varphi^{+}_{\text{in}}(t)\hat{a}^{y} - \bar{\varphi}_{\text{in}}(t)\hat{a}\big),$$

hence integrating out ϕ in the terms $S^{\text{i}}_{V}[\phi, \varphi_{\text{in}}]$ and $S^{\text{i}}_{S}[\phi]$ results in a factor one. The integral cannot be explicitly solved using the formula for the Gaussian integral since the system Hamiltonian H_S is unspecified at this point.
As it is customary in equilibrium quantum field theory, to access physical expectation values we introduce source fields into the expression of the partition function. These source fields are entered in such a way that functional derivatives of the partition function with respect to the source fields then generate the expectation values of the fields in question [1, 10]. In order to get access to the statistics of the input field, we introduce the moment generating functional,

$$\Lambda_{\text{in}}[\chi, \chi^{0}] = \int \mathcal{D}[\varphi_{\text{in}}] e^{iS^{\text{i}}_{B}[\varphi_{\text{in}}] \quad i\int_{t_0}^{t_N} dt\left[\chi(t)\varphi^{+}_{\text{in}}(t) + \chi^{0}(t)\bar{\varphi}_{\text{in}}(t)\right]}, \tag{1.108}$$

where we entered the field of the output and its hermitian conjugate on the forward and backward time branch respectively so that they generate time–ordered and normal–ordered expectation values. This functional generates the moments of the input field,

$$\left(\prod_l i\frac{\delta}{\delta\chi(t_l)}\right)\left(\prod_p i\frac{\delta}{\delta\chi^0(t_p)}\right)\Lambda_{\rm in}[\chi,\chi^0]\Big|_{\chi=\chi^0=0} = \left\langle\left[\prod_p \hat{b}^{y}_{\rm in}(t_p)\right]\left[\prod_l \hat{b}_{\rm in}(t_l)\right]\right\rangle. \tag{1.109}$$

An equivalent way to access the statistics of the input field is through the cumulant generating functional $\mathcal{S}$,

$$\mathcal{S}_{\rm in}[\chi,\chi^0] = \log\left(\Lambda_{\rm in}[\chi,\chi^0]\right). \tag{1.110}$$

This functional is defined as the logarithm of the moment generating functional Λ and as its name states, generating the cumulants of the field it describes,

$$\left(\prod_l i\frac{\delta}{\delta\chi(t_l)}\right)\left(\prod_p i\frac{\delta}{\delta\chi^0(t_p)}\right)\mathcal{S}_{\rm in}[\chi,\chi^0]\Big|_{\chi=\chi^0=0} = \left\langle\left\langle\left[\prod_p \hat{b}^{y}_{\rm in}(t_p)\right]\left[\prod_l \hat{b}_{\rm in}(t_l)\right]\right\rangle\right\rangle. \tag{1.111}$$

We will stick to the moment generating functional in the following but we want to stress that these two functionals are on equal fotting with respect to the task of determining the statistics of the output field since the cumulants can be determined from the moments and vice versa. To perfom the integration over $\varphi_{\rm in}$ we again employ the formula for the multidimensional Gaussian integral. The discrete result can be found in the appendix, see A.2.5, in the continuum limit we find,

$$\Lambda_{\rm in}[\chi,\chi^0] = e^{\int_{t_0}^{t_N} dt\left[i\chi(t)f(t)+i\chi^0(t)\bar{f}(t)+\chi(t)\chi^0(t)n_B\right]}. \tag{1.112}$$

This produces the following moments,

$$\langle\hat{b}_{\rm in}(t)\rangle = i\frac{\delta}{\delta\chi(t)}\Lambda[\chi^0,\chi]\Big|_{\chi=\chi^0=0} = f(t), \tag{1.113}$$

$$\langle\hat{b}^{y}_{\rm in}(t)\rangle = \bar{f}(t), \tag{1.114}$$

$$\langle\hat{b}^{y}_{\rm in}(t)\hat{b}_{\rm in}(t^0)\rangle = \bar{f}(t)f(t^0) + n_B\delta(t-t^0). \tag{1.115}$$

The delta distributions appearing here are a consequence of the Markov approximation we made when considering the system–bath interaction.

$g^{(2)}$-function

We define the g^2–function for different times,

$$g^{(2)}_{\rm in}(t_1,t_2,t_3,t_4,t_5,t_6) = \frac{\langle\hat{b}^{y}_{\rm in}(t_1)\,\hat{b}^{y}_{\rm in}(t_2)\,\hat{b}_{\rm in}(t_3)\,\hat{b}_{\rm in}(t_4)\rangle}{\langle\hat{b}^{y}_{\rm in}(t_5)\,\hat{b}_{\rm in}(t_6)\rangle^2},\quad t_i\in[t_0,t_N]\forall i, \tag{1.116}$$

which for the specific choice of,

$$t_1\to t,\qquad t_2\to t+\tau,\qquad t_3\to t+\tau,\qquad t_4\to t,\qquad t_5\to t,\qquad t_6\to t, \tag{1.117}$$

reduces to the usual $g^{(2)}(t,\tau)$-function. For the input field we get the following results for the constituents of the g^2–function

$$
\begin{aligned}
\langle \hat{b}^{y}_{\mathrm{in}}(t_1)\, \hat{b}^{y}_{\mathrm{in}}(t_2)\, \hat{b}_{\mathrm{in}}(t_3)\, \hat{b}_{\mathrm{in}}(t_4)\rangle = {}& \bar{f}(t_1)\bar{f}(t_2)f(t_3)f(t_4) \\
&+ n_B \left[\bar{f}(t_1)\delta(t_2-t_3)f(t_4) + \bar{f}(t_1)\delta(t_2-t_4)f(t_3)\right. \\
&\quad \left. +\bar{f}(t_2)\delta(t_1-t_3)f(t_4) + \bar{f}(t_2)\delta(t_1-t_4)f(t_3)\right] \\
&+ n_B^2[\delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3)],
\end{aligned}
\tag{1.118}
$$

$$
\langle \hat{b}^{y}_{\mathrm{in}}(t_5)\, \hat{b}_{\mathrm{in}}(t_6)\rangle^2 = \left(\bar{f}(t_5)f(t_6) + n_B\delta(t_5-t_6)\right)^2. \tag{1.119}
$$

P-functional

The P-function, sometimes referred to as the Glauber–Sudarshan P representation, is a well-known tool from quantum optics used to write down the phase space distribution of a quantum system. It is one of many formally equivalent quasi–probability distributions that describe light in the phase space formulation of quantum optics and can be generalized to multiple times in form of the P-functional [8, 9]. We can obtain the P-functional of the output field in our treatment by Fourier transforming the generating functional Λ,

$$
\begin{aligned}
P_{\mathrm{in}}[\alpha] &= \int \mathcal{D}[\chi] e^{i\int_{t_0}^{t_N} dt[\bar{\chi}(t)\alpha(t)+\bar{\alpha}(t)\chi(t)]}\Lambda_{\mathrm{in}}[\bar{\chi},\chi] \\
&= \int \mathcal{D}[\chi] e^{i\int_{t_0}^{t_N} dt\left[\bar{\chi}(t)\alpha(t)+\bar{\alpha}(t)\chi(t) \quad \bar{\chi}(t)f(t) \quad \chi(t)\bar{f}(t)+ij\chi(t)j^2 n_B\right]} \\
&= \frac{1}{n_B^N}\exp\left\{\int_{t_0}^{t_N} dt\frac{|\alpha(t)-f(t)|^2}{n_B}\right\}.
\end{aligned}
\tag{1.120}
$$

Where we employed the formula for the Gaussian integral in the last step. Note that the argument α here has nothing to do with the coherent input states. With the differential elements given as follows,

$$
\mathcal{D}[\chi] = \prod_{j=0}^{N\;\;1} \delta t d[\chi(t)], \qquad\qquad \mathcal{D}[\alpha] = \prod_{j=0}^{N\;\;1} \delta t d[\alpha(t)], \tag{1.121}
$$

the P-functional is normalized to 1.

1.4.6 Statistics of the Output Field

Similarly to the previous subsection, we want to gain access to the statistics of the output field and to that end introduce a generating functional. The discrete version of the generating functional of the output field and its derivation can be found in the appendix, see A.2.6, here we focus on the continuous limit,

$$
\Lambda_{\mathrm{out}}[\chi,\chi^{0}] = \int \mathcal{D}[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}] e^{iS^{\mathrm{io}[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}]} \quad i\int_{t_0}^{t_N} dt\left[\chi(t)\varphi^{+}_{\mathrm{out}}(t)+\chi^{0}(t)\bar{\varphi}_{\mathrm{out}}(t)\right]}. \tag{1.122}
$$

This generating functional again generates the following normal and time–ordered moments,

$$
\left(\prod_l i\frac{\delta}{\delta\chi(t_l)}\right)\left(\prod_p i\frac{\delta}{\delta\chi^{0}(t_p)}\right)\Lambda_{\mathrm{out}}[\chi,\chi^{0}]\Bigg|_{\chi=\chi^{0}=0} = \left\langle\left[\prod_p \hat{b}^{y}_{\mathrm{out}}(t_p)\right]\left[\prod_l \hat{b}_{\mathrm{out}}(t_l)\right]\right\rangle. \tag{1.123}
$$

With this we can evaluate the different moments of the output field and its hermitian conjugate as well as other common statistical quantities such as the $g^{(1)}$ or $g^{(2)}$ functions.

We integrate out the output bath modes using the multidimensional Gaussian integral. In the exponent we find,

$$
\begin{aligned}
iS = iS_S^{\mathrm{io}}[\phi] - &\int_{t_0}^{t_N} dt\big(\bar{\varphi}_{\mathrm{in}}^{+}(t)\ \bar{\varphi}_{\mathrm{out}}^{+}(t)\ \bar{\varphi}_{\mathrm{out}}(t)\ \bar{\varphi}_{\mathrm{in}}(t)\big)\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} \varphi_{\mathrm{in}}^{+}(t) \\ \varphi_{\mathrm{out}}^{+}(t) \\ \varphi_{\mathrm{out}}(t) \\ \varphi_{\mathrm{in}}(t) \end{pmatrix} \\
&- \sqrt{\kappa}\int_{t_0}^{t_N} dt\big[\bar{\phi}^{+}(t)\varphi_{\mathrm{in}}^{+} - \bar{\varphi}_{\mathrm{out}}^{+}(t)\phi^{+}(t) - \bar{\phi}\ \ (t)\varphi_{\mathrm{out}}(t) + \bar{\varphi}_{\mathrm{in}}(t)\phi\ \ (t)\big] \\
&- i\int_{t_0}^{t_N} dt\big[\chi(t)\varphi_{\mathrm{out}}^{+}(t) + \chi^{0}(t)\bar{\varphi}_{\mathrm{out}}(t)\big].
\end{aligned}
\tag{1.124}
$$

Evaluating the path integral over the output modes leads to the following action,

$$
S[\phi, \varphi_{\mathrm{in}}, \chi, \chi^{0}] = S^{\mathrm{i}}[\phi, \varphi_{\mathrm{in}}] - \int_{t_0}^{t_N} dt\big[\chi(t)\big(\varphi_{\mathrm{in}}^{+}(t) + \sqrt{\kappa}\phi^{+}(t)\big) - \chi^{0}(t)\big(\bar{\varphi}_{\mathrm{in}}(t) + \sqrt{\kappa}\bar{\phi}\ \ (t)\big)\big].
\tag{1.125}
$$

Integrating out the input fields as well and using as an input state the displaced thermal state we get,

$$
\begin{aligned}
S[\phi, \chi, \chi^{0}] =& S[\phi] - \int_{t_0}^{t_N} dt\chi(t)\left[f(t) + \sqrt{\frac{\kappa}{2}}(2n_B+1)\phi^{\mathrm{q}}(t) + \sqrt{\frac{\kappa}{2}}\phi^{\mathrm{cl}}(t)\right] \\
&- \int_{t_0}^{t_N} dt\chi^{0}(t)\left[\bar{f}(t) - \sqrt{\frac{\kappa}{2}}(2n_B+1)\bar{\phi}^{\mathrm{q}}(t) + \sqrt{\frac{\kappa}{2}}\bar{\phi}^{\mathrm{cl}}(t)\right] + i\int_{t_0}^{t_N} \chi(t)\chi^{0}(t)n_B.
\end{aligned}
\tag{1.126}
$$

Here $S[\phi]$ is known from the section on the reduced action and can be found in Eq. (1.94). This leaves us with the moment generating function now only being a path integral over the system modes ϕ,

$$
\boxed{\Lambda_{\mathrm{out}}[\chi, \chi^{0}] = \int \mathcal{D}[\phi]e^{iS[\phi,\chi,\chi^{0}]}.}
\tag{1.127}
$$

The action here is,

$$
\boxed{
\begin{aligned}
S[\phi, \chi, \chi^{0}] =& S_S[\phi] + i\kappa\int_{t_0}^{t_N} dt\left[(2n_B+1)\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{q}}(t) - \frac{1}{2}\bar{\phi}^{\mathrm{cl}}(t)\phi^{\mathrm{q}}(t) + \frac{1}{2}\bar{\phi}^{\mathrm{q}}(t)\phi^{\mathrm{cl}}(t)\right] \\
&+ i\sqrt{2\kappa}\int_{t_0}^{t_N} dt\big[f(t)\bar{\phi}^{\mathrm{q}}(t) - \bar{f}(t)\phi^{\mathrm{q}}(t)\big] \\
&- \int_{t_0}^{t_N} dt\chi(t)\left[f(t) + \sqrt{\frac{\kappa}{2}}(2n_B+1)\phi^{\mathrm{q}}(t) + \sqrt{\frac{\kappa}{2}}\phi^{\mathrm{cl}}(t)\right] \\
&- \int_{t_0}^{t_N} dt\chi^{0}(t)\left[\bar{f}(t) - \sqrt{\frac{\kappa}{2}}(2n_B+1)\bar{\phi}^{\mathrm{q}}(t) + \sqrt{\frac{\kappa}{2}}\bar{\phi}^{\mathrm{cl}}(t)\right] \\
&+ i\int_{t_0}^{t_N} \chi(t)\chi^{0}(t)n_B.
\end{aligned}
}
\tag{1.128}
$$

Note that this, with the exception of the input state which we chose to be a displaced thermal state, is still general and $S_S[\phi]$ is the only system specific term.

P-functional

We can obtain the P–functional by Fourier transforming the generating functional Λ,

$$P_{\text{out}}[\alpha] = \int \mathcal{D}[\chi] e^{i\int_{t_0}^{t_N} dt[\bar{\chi}(t)\alpha(t)+\bar{\alpha}(t)\chi(t)]} \Lambda_{\text{out}}[\bar{\chi},\chi] = \frac{1}{n_B^N}\int \mathcal{D}[\phi] e^{iS_P[\phi,\alpha]}. \tag{1.129}$$

With

$$\begin{aligned} S_P[\phi,\alpha] = S[\phi] + i\int_{t_0}^{t_N} \frac{dt}{n_B} &\left[\bar{\alpha}(t) - \bar{f}(t) + \sqrt{\frac{\kappa}{2}}(2n_B+1)\bar{\phi}^{\text{q}}(t) - \sqrt{\frac{\kappa}{2}}\bar{\phi}^{\text{cl}}(t)\right] \\ &\left[\alpha(t) - f(t) - \sqrt{\frac{\kappa}{2}}(2n_B+1)\phi^{\text{q}}(t) - \sqrt{\frac{\kappa}{2}}\phi^{\text{cl}}(t)\right], \end{aligned} \tag{1.130}$$

and

$$\begin{aligned} S[\phi] = S_S[\phi] &+ i\sqrt{2\kappa}\int_{t_0}^{t_N} dt\big(f(t)\bar{\phi}^{\text{q}}(t) - \bar{f}(t)\phi^{\text{q}}(t)\big) \\ &+ i\kappa\int_{t_0}^{t_N} dt\Big[(2n_B+1)\bar{\phi}^{\text{q}}(t)\phi^{\text{q}}(t) - \frac{1}{2}\bar{\phi}^{\text{cl}}(t)\phi^{\text{q}}(t) + \frac{1}{2}\bar{\phi}^{\text{q}}(t)\phi^{\text{cl}}(t)\Big]. \end{aligned} \tag{1.131}$$

The discrete version is given as follows,

$$\begin{aligned} S_P[\phi,\alpha] = S[\phi] + i\sum_{j=0}^{N\ 1} \frac{\delta t}{n_B} &\left[\bar{\alpha}_j - \bar{f}_j + \sqrt{\frac{\kappa}{2}}(2n_B+1)\bar{\phi}_j^{\text{q}} - \sqrt{\frac{\kappa}{2}}\bar{\phi}_j^{\text{cl}}\right] \\ &\left[\alpha_j - f_j - \sqrt{\frac{\kappa}{2}}(2n_B+1)\phi_j^{\text{q}} - \sqrt{\frac{\kappa}{2}}\phi_j^{\text{cl}}\right], \end{aligned} \tag{1.132}$$

with

$$\begin{aligned} S[\phi] = S_S[\phi] \quad &+ i\frac{\kappa}{2}\sum_{j=1}^{N\ 1} \delta t\big[\bar{\phi}_{j+1}^{+}\phi_{j\ 1}^{+} + \bar{\phi}_{j\ 1}\phi_{j+1} - 2\bar{\phi}_{j\ 1}\phi_{j+1}^{+}\big] \\ &+ \delta t\sqrt{\kappa}\sum_{j=1}^{N\ 2} \big[f_j\big(\bar{\phi}_{j\ 1} - \bar{\phi}_{j+1}^{+}\big) + \bar{f}_j\big(\phi_{j\ 1}^{+} + \phi_{j+1}\big)\big] \\ &+ \delta t\kappa n_B\sum_{j=1}^{N\ 2} \big(\bar{\phi}_{j\ 1} - \bar{\phi}_{j+1}^{+}\big)\big(\phi_{j\ 1}^{+} - \phi_{j+1}\big). \end{aligned} \tag{1.133}$$

Chapter 2

Damped Harmonic Oscillator

In this chapter we apply our novel formalism to a simple system which is solvable through the methods of standard input–output theory as well. This serves as a way to familiarize ourselves with the formalism and as a proof of concept. The system we consider is that of a single mode which is coupled to a Markovian bath which results in a damped harmonic oscillator. We will first solve for the statistics of the input and output field using standard input–output theory and then show how the same results can be obtained using the Keldysh path integral approach to input–output theory.

2.1 Damped Harmonic Oscillator with Standard Input-Output Theory

The Hamiltonian governing our system has the following form,

$$H_S = \omega_S \hat{a}^\dagger \hat{a}, \tag{2.1}$$

where the system mode $\hat{a}$ obeys the following commutation relation,

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{2.2}$$

For this specific Hamiltonian the equation of motion of the system mode from Eq. (1.23) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = i\omega_S\left[\hat{a}^\dagger\hat{a}, \hat{a}\right] - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t}) = -i\omega_S\hat{a} - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t}) = -\Big(i\omega_S + \frac{\kappa}{2}\Big)\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{in}}(\mathrm{t})\,. \tag{2.3}$$

Equivalently we find the equation of motion containing the output field,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\Big(i\omega_S - \frac{\kappa}{2}\Big)\hat{a} - \sqrt{\kappa}\,\hat{b}_{\mathrm{out}}(\mathrm{t})\,. \tag{2.4}$$

We can formally solve these equations to find,

$$\hat{a}(t) = e^{-(i\omega_S + \frac{\kappa}{2})(t - t_0)}\hat{a}(t_0) - \sqrt{\kappa}\int_{t_0}^{t} d\tau\, e^{-(i\omega_S + \frac{\kappa}{2})(t-\tau)}\hat{b}_{\mathrm{in}}(\tau). \tag{2.5}$$

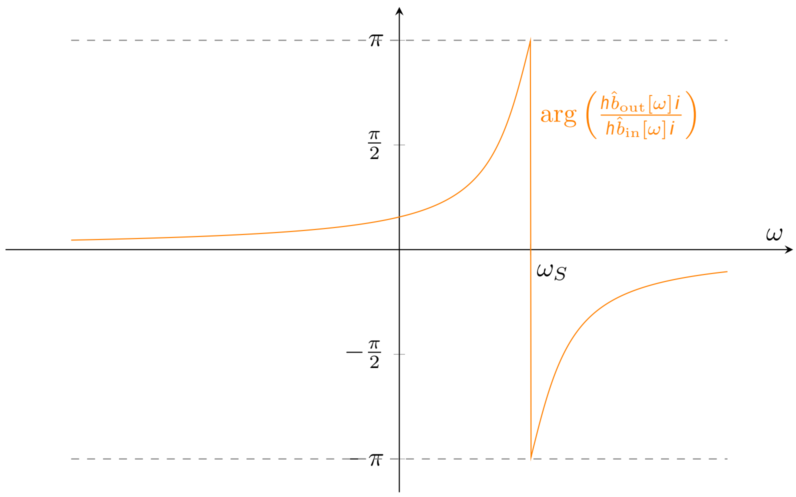

Figure 2.1: Plot of the phase of the transmission coefficient for fixed κ. The line connecting the minimal and maximal value of the transmission coefficient in this plot is an artefact.

This represents damped motion for $\langle \hat{a} \rangle$ even with no input, $\langle \hat{b}_{\text{in}}(\text{t}) \rangle = 0$, since $\hat{a}$ oscillates at the frequency ω_S and contains the exponential damping factor $e^{\ \frac{\kappa}{2}t}$. In the limit of no coupling, i.e. $\kappa \to 0$, we retrieve the unitary time-evolution,

$$\hat{a}(t) = e^{\ i\omega_S(t \quad t_0)} \hat{a}(t_0). \tag{2.6}$$

In the following we reproduce a derivation of the relationship between the input and output fields given in [5]. We start by Fourier transforming the equations of motion of the system mode to get,

$$-i\omega \hat{a}[\omega] = -\left(i\omega_S + \frac{\kappa}{2}\right)\hat{a}[\omega] - \sqrt{\kappa}\hat{b}_{\text{in}}[\omega], \tag{2.7}$$

$$-i\omega \hat{a}[\omega] = -\left(i\omega_S - \frac{\kappa}{2}\right)\hat{a}[\omega] - \sqrt{\kappa}\hat{b}_{\text{out}}[\omega]. \tag{2.8}$$

For the convention of the Fourier transform in use here, check B.1. Combining these equations leads to the following relation between the input and the output field,

$$\hat{b}_{\text{out}}[\omega] = \frac{2(\omega_S - \omega) + i\kappa}{2(\omega_S - \omega) - i\kappa}\hat{b}_{\text{in}}[\omega]. \tag{2.9}$$

With this we recover the result from [3] and from this expression we can read off the reflection coefficient,

$$R[\omega] = \frac{\langle \hat{b}_{\text{out}}[\omega] \rangle}{\langle \hat{b}_{\text{in}}[\omega] \rangle},$$

see Fig. 2.1.

Furthermore we can rearrange Eq. (2.7) to get the following relationship between the input field and the system mode,

$$\hat{a}[\omega] = \frac{-\sqrt{\kappa}}{\frac{\kappa}{2} + i(\omega_S - \omega)}\hat{b}_{\text{in}}[\omega]. \tag{2.10}$$

We now consider the case of resonant driving, i.e. $\omega_S = \omega$. In this specific case the relations derived above simplify as follows,

$$\hat{b}_{\text{out}}[\omega_S] = -\hat{b}_{\text{in}}[\omega_S], \tag{2.11}$$

$$\hat{a}[\omega_S] = -\frac{2}{\sqrt{\kappa}}\hat{b}_{\text{in}}[\omega_S] = \frac{2}{\sqrt{\kappa}}\hat{b}_{\text{out}}[\omega_S]. \tag{2.12}$$

Displaced Thermal State as Input

So far our treatment was independent of the specific input state. Now we assume that the bath is initially in a displaced thermal state,

$$\hat{\rho}_B = \bigotimes_k \left(1 - e^{-\beta\omega_k}\right)\hat{D}(\alpha_k)e^{-\beta\omega_k \hat{b}_k^\dagger \hat{b}_k}\hat{D}^\dagger(\alpha_k). \tag{2.13}$$

For this specific scenario we can explicitly compute the moments of the input field,

$$\begin{aligned}
\langle \hat{b}_{\text{in}}(t)\rangle = \text{Tr}\left\{\hat{b}_{\text{in}}(t)\hat{\rho}_B\right\} &= \frac{1}{2\pi\rho}\sum_k e^{-i\omega_k(t-t_0)}\langle \hat{b}_k(t_0)\rangle \\
&= \frac{1}{2\pi\rho}\sum_k e^{-i\omega_k(t-t_0)}\alpha_k = \sum_k e^{-i\omega_k(t-t_0)}\frac{\alpha_k}{\sqrt{N\delta t}} = f(t), \qquad (2.14)\\
\langle \hat{b}_{\text{in}}^\dagger(t)\rangle &= \overline{\langle \hat{b}_{\text{in}}(t)\rangle} = \bar{f}(t), \qquad (2.15)\\
\langle \hat{b}_{\text{in}}^\dagger(t)\,\hat{b}_{\text{in}}(t')\rangle &= \bar{f}(t)f(t') + \delta(t-t')n_B, \qquad (2.16)
\end{aligned}$$

where we assumed, that $e^{-\beta\omega_k} \approx e^{-\beta\Omega}$ and then attain the same results that we get from the Keldysh approach in Eqs. (1.113), (1.115).
We can further compute the moments of the output field by using the input–output relation from Eq. (1.26),

$$\langle \hat{b}_{\text{out}}(t)\rangle = \langle \hat{b}_{\text{in}}(t)\rangle + \sqrt{\kappa}e^{-i\omega_S(t-t_0)}e^{-\kappa/2(t-t_0)}\langle \hat{a}(t_0)\rangle - \kappa\int_{t_0}^{t} d\tau\, e^{-i\omega_S(t-\tau)}e^{-\kappa/2(t-\tau)}\langle \hat{b}_{\text{in}}(\tau)\rangle, \tag{2.17}$$

which in the limit where the initial time is in the distant past, $t_0 \to -\infty$, simplifies to the following,

$$\langle \hat{b}_{\text{out}}(t)\rangle = f(t) - i\kappa\int dt'\, f(t')G^R(t-t'), \tag{2.18}$$

with,

$$G^R(t-t') = -i\theta(t-t')e^{-i\omega_S(t-t')}e^{-\kappa/2(t-t')}. \tag{2.19}$$

In the style of the Keldysh formalism we introduced the retarded Green function $G^R(t-t')$ here. For the hermitian conjugate of the above we will use the advanced Green function G^A which is connected to the retarded Green function in the following way,

$$\left[G^A(t-t')\right]^\dagger = G^R(t'-t), \tag{2.20}$$

note that throughout this thesis, the single Green functions are always considered in the continuum limit and are therefore not matrices. This leads to,

$$\langle \hat{b}^{\dagger}_{\text{out}}(t)\rangle = \bar{f}(t) + i\kappa \int dt' \, \bar{f}(t')G^A(t'-t). \tag{2.21}$$

Further we can compute the output flux,

$$\begin{aligned}
\langle \hat{b}^{\dagger}_{\text{out}}(t_1)\, \hat{b}_{\text{out}}(t_2)\rangle =& \bar{f}(t_1)f(t_2) + \delta(t_1-t_2)n_B \\
&- i\kappa\left(n_B\big(G^R(t_2-t_1) - G^A(t_2-t_1)\big)\right. \\
&\qquad \left. + \bar{f}(t_1)\int dt'\, G^R(t_2-t')f(t') - f(t_2)\int dt'\, G^A(t'-t_1)\bar{f}(t')\right) \\
&+ \kappa^2\left(-n_B\int dt'\, G^A(t'-t_1)G^R(t_2-t')\right. \\
&\qquad \left. + \int dt \int dt'\, \bar{f}(t')G^A(t'-t_1)G^R(t_2-t)f(t)\right) \\
=& \langle \hat{b}^{\dagger}_{\text{out}}(t_1)\rangle\langle \hat{b}_{\text{out}}(t_2)\rangle + n_B\delta(t_1-t_2),
\end{aligned} \tag{2.22}$$

which in the limit of zero temperature, $n_B \to 0$, simplifies to the absolute value squared of the average output field.

2.2 Damped Harmonic Oscillator with Keldysh Input–Output Theory

We now turn to the treatment of the damped harmonic oscillator with the Keldysh path integral approach detailed in Sec. 1.3. The central goal will be to evaluate the moment generating functional from Eq. (1.127) to get access to the statistics of the output field,

$$\begin{aligned}
\Lambda_{\text{out}}[\chi,\chi'] &= \int \mathcal{D}[\phi,\varphi_{\text{in}},\varphi_{\text{out}}]e^{iS^{\text{io}[\phi,\varphi_{\text{in}},\varphi_{\text{out}}]} - i\int_{t_0}^{t_N} dt\left[\chi(t)\varphi^{+}_{\text{out}}(t) + \chi'(t)\bar{\varphi}^{-}_{\text{out}}(t)\right]} \\
&= \int \mathcal{D}[\phi]e^{iS[\phi,\chi,\chi']}.
\end{aligned} \tag{2.23}$$

With the system Hamiltonian $H_S = \omega_S \hat{a}^{\dagger}\hat{a}$, we can evaluate the system specific contribution $S_S[\phi]$ to the action,

$$\begin{aligned}
S_S[\phi] =& \sum_{j=1}^{N-1}\delta t\left[i\bar{\phi}^{+}_{j}\frac{\phi^{+}_{j}-\phi^{+}_{j-1}}{\delta t} - \omega_S\bar{\phi}^{+}_{j}\phi^{+}_{j-1}\right] \\
&- \sum_{j=1}^{N-1}\delta t\left[i\bar{\phi}^{-}_{j}\frac{\phi^{-}_{j}-\phi^{-}_{j-1}}{\delta t} - \omega_S\bar{\phi}^{-}_{j}\phi^{-}_{j-1}\right] \\
&+ i\bar{\phi}_0\phi_0 - i\ln\rho_S(\bar{\phi}_0,\phi_{2N-1}).
\end{aligned} \tag{2.24}$$

Neglecting the boundary terms and going to the continuum limit we find,

$$\begin{aligned}
S_S[\phi] &= \int_{t_0}^{t_N} dt\left[\bar{\phi}^{+}(t)(i\partial_t - \omega_S)\phi^{+}(t) - \bar{\phi}^{-}(t)(i\partial_t-\omega_S)\phi^{-}(t)\right] \\
&= \int_{t_0}^{t_N} dt\left[\bar{\phi}^{\text{q}}(t)(i\partial_t - \omega_S)\phi^{\text{cl}}(t) + \bar{\phi}^{\text{cl}}(t)(i\partial_t - \omega_S)\phi^{\text{q}}(t)\right].
\end{aligned} \tag{2.25}$$

Inserting this back into the action $S[\phi,\chi,\chi^0]$ we get,

$$S[\phi,\chi,\chi^0] = \int_{t_0}^{t_N} dt \begin{pmatrix} \bar{\phi}^{\mathrm{cl}}(t) \\ \bar{\phi}^{\mathrm{q}}(t) \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{i\kappa}{2} + (i\partial_t - \omega_S) \\ \frac{i\kappa}{2} + (i\partial_t - \omega_S) & i\kappa(2n_B+1) \end{pmatrix} \begin{pmatrix} \phi^{\mathrm{cl}}(t) \\ \phi^{\mathrm{q}}(t) \end{pmatrix} \tag{2.26}$$
$$+ \begin{pmatrix} -\chi(t)\sqrt{\frac{\kappa}{2}} \\ -i\bar{f}(t)\sqrt{2\kappa} - \chi(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) \end{pmatrix}^T \begin{pmatrix} \phi^{\mathrm{cl}}(t) \\ \phi^{\mathrm{q}}(t) \end{pmatrix}$$
$$+ \begin{pmatrix} -\chi^0(t)\sqrt{\frac{\kappa}{2}} \\ if(t)\sqrt{2\kappa} + \chi^0(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) \end{pmatrix}^T \begin{pmatrix} \bar{\phi}^{\mathrm{cl}}(t) \\ \bar{\phi}^{\mathrm{q}}(t) \end{pmatrix}$$
$$- \chi(t)f(t) - \chi^0(t)\bar{f}(t) + in_B\chi(t)\chi^0(t).$$

We now artificially introduce a second integral over time t^0 for the quadratic term in order to cast the path integral into the general Gaussian form to which we can apply the formula for multi-dimensional Gaussian integrals from A.2.1,

$$S[\phi,\chi,\chi^0] = \tag{2.27}$$
$$\left[\int_{t_0}^{t_N} dt^0 \int_{t_0}^{t_N} dt \begin{pmatrix} \bar{\phi}^{\mathrm{cl}}(t) \\ \bar{\phi}^{\mathrm{q}}(t) \end{pmatrix}^T \begin{pmatrix} 0 & (-\frac{i\kappa}{2} + (i\partial_t - \omega_S))\delta(t-t^0) \\ (\frac{i\kappa}{2} + (i\partial_t - \omega_S))\delta(t-t^0) & i\kappa(2n_B+1)\delta(t-t^0) \end{pmatrix} \begin{pmatrix} \phi^{\mathrm{cl}}(t^0) \\ \phi^{\mathrm{q}}(t^0) \end{pmatrix}\right]$$
$$+ \int_{t_0}^{t_N} dt \begin{pmatrix} -\chi(t)\sqrt{\frac{\kappa}{2}} \\ -i\bar{f}(t)\sqrt{2\kappa} - \chi(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) \end{pmatrix}^T \begin{pmatrix} \phi^{\mathrm{cl}}(t) \\ \phi^{\mathrm{q}}(t) \end{pmatrix} + \begin{pmatrix} -\chi^0(t)\sqrt{\frac{\kappa}{2}} \\ if(t)\sqrt{2\kappa} + \chi^0(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) \end{pmatrix}^T \begin{pmatrix} \bar{\phi}^{\mathrm{cl}}(t) \\ \bar{\phi}^{\mathrm{q}}(t) \end{pmatrix}$$
$$- \chi(t)f(t) - \chi^0(t)\bar{f}(t) + i\chi(t)\chi^0(t).$$

From this expression we read of the inverse Green functions,

$$\left[G\ \ 1\right]^A(t-t^0) = \left(-\frac{i\kappa}{2} + (i\partial_t - \omega_S)\right)\delta(t-t^0), \tag{2.28}$$
$$\left[G\ \ 1\right]^R(t-t^0) = \left(\frac{i\kappa}{2} + (i\partial_t - \omega_S)\right)\delta(t-t^0), \tag{2.29}$$
$$\left[G\ \ 1\right]^K(t-t^0) = i\kappa(2n_B+1)\delta(t-t^0). \tag{2.30}$$

By Fourier transforming and solving for the inverse, we determine the Green functions in the frequency domain,

$$G^A[\omega] = \frac{1}{\omega - \omega_S - {}^{i\kappa}/_2}, \tag{2.31}$$
$$G^R[\omega] = \frac{1}{\omega - \omega_S + {}^{i\kappa}/_2}, \tag{2.32}$$
$$G^K[\omega] = -G^R[\omega] \cdot \left[G\ \ 1\right]^K[\omega] \cdot G^A[\omega], \qquad [G\ \ 1]^K[\omega] = i\kappa(2n_B+1)). \tag{2.33}$$

Note that already at this stage we find that the relation of $G^R[\omega]$ to $G^A[\omega]$ is the same as the relation between output and input in the classical input-output theory, see Eq. (2.9),

$$G^R[\omega] = \frac{2(\omega_S - \omega) + i\kappa}{2(\omega_S - \omega) - i\kappa} G^A[\omega]. \tag{2.34}$$

The Green functions in the time domain can now be found by a further Fourier transform,

$$G^A(t-t^0) = i\theta(t^0 - t)e^{\ \ i\omega_S(t\ \ t^0)}e^{\kappa/2(t\ \ t^0)}, \tag{2.35}$$
$$G^R(t-t^0) = -i\theta(t-t^0)e^{\ \ i\omega_S(t\ \ t^0)}e^{\ \ \kappa/2(t\ \ t^0)}, \tag{2.36}$$
$$G^K(t-t^0) = -iFe^{\ \ i\omega_S(t\ \ t^0)}e^{\ \ \kappa/2jt\ \ t^0j},$$

where we introduced the distribution function $F = 2n_B + 1$. With the Green functions we can now define $\underline{G}$,

$$
\begin{aligned}
\underline{G}(t-t') &= \begin{pmatrix} G^K(t-t') & G^R(t-t') \\ G^A(t-t') & 0 \end{pmatrix} \\
&= \begin{pmatrix} -iFe^{-i\omega_S(t-t')}e^{-\kappa/2|t-t'|} & -i\theta(t-t')e^{-i\omega_S(t-t')}e^{-\kappa/2(t-t')} \\ i\theta(t'-t)e^{-i\omega_S(t-t')}e^{\kappa/2(t-t')} & 0 \end{pmatrix},
\end{aligned}
\tag{2.37}
$$

and with it evaluate the moment generating function,

$$
\Lambda_{\text{out}}[\chi, \chi'] = e^{iS[\chi,\chi']}. \tag{2.38}
$$

If we let our dynamics start in the far past, $t_0 \to -\infty$, and end in the far future, $t_N \to \infty$, we find,

$$
\begin{aligned}
S[\chi, \chi'] = &\int dt \int dt'\, i\kappa\big(\chi(t)G^R(t-t')f(t') - \bar{f}(t)G^A(t-t')\chi'(t')\big) \\
&- \int dt\, \big(\chi(t)f(t) + \chi'(t)\bar{f}(t) - i\chi(t)\chi'(t)n_B\big).
\end{aligned}
\tag{2.39}
$$

From the moment generating functional we now have access to the statistics of the output field. We start by calculating the first moment of the output field and its hermitian conjugate,

$$
\langle \hat{b}_{\text{out}}(t)\rangle = i\, \frac{\delta\Lambda_{\text{out}}[\chi,\chi']}{\delta\chi(t)}\bigg|_{\chi=\chi'=0} = f(t) - i\kappa \int dt'\, f(t')G^R(t-t'), \tag{2.40}
$$

$$
\langle \hat{b}^\dagger_{\text{out}}(t)\rangle = \overline{\langle \hat{b}_{\text{out}}(t)\rangle} = \bar{f}(t) + i\kappa \int dt'\, \bar{f}(t')G^A(t'-t). \tag{2.41}
$$

Through these averages we can rewrite the exponent of our generating functional in a concise manner,

$$
\begin{aligned}
\Lambda_{\text{out}}[\chi, \chi'] &= e^{iS[\chi,\chi']}, \\
S[\chi, \chi'] &= -\int dt\Big(\chi(t)\langle \hat{b}_{\text{out}}(t)\rangle + \chi'(t)\langle \hat{b}^\dagger_{\text{out}}(t)\rangle + in_B\chi(t)\chi'(t)\Big).
\end{aligned}
\tag{2.42}
$$

From here it is also convenient to compute the cumulant generating functional from Eq. (1.111),

$$
\mathcal{S}_{\text{out}}[\chi, \chi'] = -i\int dt\Big(\chi(t)\langle \hat{b}_{\text{out}}(t)\rangle + \chi'(t)\langle \hat{b}^\dagger_{\text{out}}(t)\rangle + in_B\chi(t)\chi'(t)\Big). \tag{2.43}
$$

From either of these expressions it is now straightforward to compute further moments of the output field. We firstly calculate the $g^{(1)}$–function,

$$
\begin{aligned}
\langle \hat{b}^\dagger_{\text{out}}(t_1)\hat{b}_{\text{out}}(t_2)\rangle &= (i)^2\, \frac{\delta^2\Lambda[\chi,\chi']}{\delta\chi'(t_1)\delta\chi(t_2)}\bigg|_{\chi=\chi'=0} \\
&= \bar{f}(t_1)f(t_2) + \delta(t_2-t_1)n_B \\
&\quad - i\kappa\Big(\bar{f}(t_1)\int dt'\, G^R(t_2-t')f(t') - f(t_2)\int dt'\, G^A(t'-t_1)\bar{f}(t')\Big) \\
&\quad + \kappa^2\Big(\int dt \int dt'\, f(t')\bar{f}(t)G^R(t_2-t')G^A(t-t_1)\Big), \\
&= \overline{\langle \hat{b}_{\text{out}}(t_1)\rangle}\langle \hat{b}_{\text{out}}(t_2)\rangle + \delta(t_2-t_1)n_B.
\end{aligned}
\tag{2.44}
$$

Further we calculate the $g^{(2)}$-function in the same fashion as we did for the input-field,

$$
\begin{aligned}
\langle \hat{b}^{\dagger}_{\text{out}}(t_1)\, \hat{b}^{\dagger}_{\text{out}}(t_2)\, \hat{b}_{\text{out}}(t_3)\, \hat{b}_{\text{out}}(t_4)\rangle &= (i)^4 \frac{\delta \Lambda_{\text{out}}[\chi, \chi']}{\delta\chi'(t_1)\delta\chi'(t_2)\delta\chi(t_3)\delta\chi(t_4)}\bigg|_{\chi=\chi'=0} \\
&= \overline{\langle \hat{b}_{\text{out}}(t_1)\rangle \langle \hat{b}_{\text{out}}(t_2)\rangle} \langle \hat{b}_{\text{out}}(t_3)\rangle \langle \hat{b}_{\text{out}}(t_4)\rangle \\
&- n_B \Big(\langle \hat{b}_{\text{out}}(t_4)\rangle \overline{\langle \hat{b}_{\text{out}}(t_2)\rangle} \delta(t_3 - t_1) + \langle \hat{b}_{\text{out}}(t_3)\rangle \overline{\langle \hat{b}_{\text{out}}(t_2)\rangle} \delta(t_4 - t_1) \\
&+ \langle \hat{b}_{\text{out}}(t_4)\rangle \overline{\langle \hat{b}_{\text{out}}(t_1)\rangle} \delta(t_3 - t_2) + \langle \hat{b}_{\text{out}}(t_3)\rangle \overline{\langle \hat{b}_{\text{out}}(t_1)\rangle} \delta(t_4 - t_2) \Big) \\
&+ n_B^2 (\delta(t_3 - t_2)\delta(t_4 - t_1) + \delta(t_4 - t_2)\delta(t_3 - t_1)).
\end{aligned} \tag{2.45}
$$

For the special case of no coupling between the bath and our system, $\kappa \to 0$, we find,

$$
\begin{aligned}
g^{(2)}_{\text{out}}(t_1, t_2, \ldots, t_6) \cdot \left(\bar{f}(t_5) f(t_6)\right)^2 &= \bar{f}(t_1)\bar{f}(t_2) f(t_3) f(t_4) \\
&+ n_B \left(\bar{f}(t_1)\delta(t_3 - t_2) f(t_4) + \bar{f}(t_2)\delta(t_4 - t_1) f(t_3) \right. \\
&\quad \left. + \bar{f}(t_2)\delta(t_3 - t_1) f(t_4) + \bar{f}(t_1)\delta(t_2 - t_4) f(t_3) \right) \\
&+ n_B^2 (\delta(t_2 - t_4)\delta(t_3 - t_1) + \delta(t_3 - t_2)\delta(t_4 - t_1)) \\
&= g^{(2)}_{\text{in}}(t_1, t_2, \ldots, t_6) \cdot \left(\bar{f}(t_5) f(t_6)\right)^2,
\end{aligned} \tag{2.46}
$$

as expected from the input-output relation and the $g^{(2)}$–function of the input field. In the case of zero temperature, i.e. $n_B \to 0$, we get,

$$
g^{(2)}_{\text{out}}(t_1, t_2, \ldots, t_6) = \frac{\overline{\langle \hat{b}_{\text{out}}(t_1)\rangle \langle \hat{b}_{\text{out}}(t_2)\rangle} \langle \hat{b}_{\text{out}}(t_3)\rangle \langle \hat{b}_{\text{out}}(t_4)\rangle}{\left(\overline{\langle \hat{b}_{\text{out}}(t_5)\rangle} \langle \hat{b}_{\text{out}}(t_6)\rangle \right)^2}. \tag{2.47}
$$

P-Functional

In the case of the damped harmonic oscillator it is also possible to evaluate the P–functional exactly,

$$
P_{\text{out}}[\alpha] = \int \mathcal{D}[\chi] e^{i \int_{t_0}^{t_N} dt [\bar{\chi}(t)\alpha(t) + \bar{\alpha}(t)\chi(t)]} \Lambda_{\text{out}}[\bar{\chi}, \chi] = \frac{1}{n_B^N} e^{i S_P[\alpha]}, \tag{2.48}
$$

$$
\begin{aligned}
S_P[\alpha] &= -\frac{i}{n_B} \int dt \Big(-i\kappa \int du\, G^A(u - t)\bar{f}(u) - \bar{f}(t) + \bar{\alpha}(t) \Big) \\
&\qquad \Big(i\kappa \int dv\, G^R(t - v) f(v) - f(t) + \alpha(t) \Big)
\end{aligned} \tag{2.49}
$$

$$
= -i \int dt \frac{|\alpha(t) - \langle \hat{b}_{\text{out}}(t)\rangle|^2}{n_B}. \tag{2.50}
$$

The P-functional here corresponds to the P-functional of a displaced thermal state where the displacement is given by the average of the output field $\langle \hat{b}_{\text{out}}(t)\rangle$ and it reduces to the P-functional of the input field in the limit of no coupling, $\kappa \to 0$,

$$
P_{\text{out}}[\alpha] \xrightarrow{\kappa \to 0} \frac{1}{n_B^N} e^{\int dt \frac{|\alpha(t) - f(t)|^2}{n_B}} = P_{\text{in}}[\alpha]. \tag{2.51}
$$

Note that here we also made the same assumptions on the initial and final times as made for the generating functional above, $t_0 \to -\infty$, $t_N \to \infty$.

2.2.1 Frequency Space

We now switch to describing the dynamics of our system in the frequency domain. This is achieved by rewriting the generating functional from Eq. (2.38) such that it generates the moments of the output field in frequency space and a subsequent derivation of the statistical quantities in question via functional derivatives as before. We consider the exponent of the generating functional in Eq. (2.23) where we introduced the source fields in the time-domain,

$$iS^{\mathrm{io}}[\phi, \varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}] - i \int_{t_0}^{t_N} dt \big[\chi(t)\varphi^{+}_{\mathrm{out}}(t) + \chi^{0}(t)\bar{\varphi}_{\mathrm{out}}(t)\big] = \dots, \tag{2.52}$$

and rewrite the fields $\varphi^{+}_{\mathrm{out}}(t), \bar{\varphi}_{\mathrm{out}}(t)$ using their Fourier transforms, assuming $t_0 \to -\infty, t_N \to \infty$,

$$\dots = iS^{\mathrm{io}}[\phi, \varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}] - \frac{i}{2\pi} \int_{1}^{1} d\omega \big(\varphi^{+}_{\mathrm{out}}[\omega]\bar{\chi}[\omega] + \bar{\varphi}_{\mathrm{out}}[\omega]\chi^{0}[\omega]\big). \tag{2.53}$$

From this expression we can see that functional derivatives with respect to $\bar{\chi}$ will generate the moments of the output field in frequency space and functional derivatives with respect to χ^{0} will generate the moments of the hermitian conjugate once the functional has been rewritten through the Fourier transforms of its constituens. This means in frequency space we replace $\Lambda_{\mathrm{out}}[\chi, \chi^{0}] \to \Lambda_{\mathrm{out}}[\bar{\chi}, \chi^{0}]$. This reformulation leads to

$$S[\bar{\chi}, \chi^{0}] = -\frac{1}{2\pi} \int_{1}^{1} d\omega \, i\kappa \left(\bar{f}[\omega]G^{A}[\omega]\chi^{0}[\omega] - \bar{\chi}[\omega]G^{R}[\omega]f[\omega]\right) + \bar{\chi}[\omega]f[\omega] \tag{2.54}$$

$$+\chi^{0}[\omega]\bar{f}[\omega] - in_{B}\bar{\chi}[\omega]\chi^{0}[\omega]\big) . \tag{2.55}$$

Similarly to the procedure in the time domain we compute the first moment and its hermitian conjugate,

$$\left.\frac{\delta\Lambda_{\mathrm{out}}[\bar{\chi}, \chi^{0}]}{\delta\bar{\chi}[\omega]}\right|_{\bar{\chi}=\chi^{0}=0} = -\frac{i}{2\pi} \int \mathcal{D}[\phi, \varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}]\varphi^{+}_{\mathrm{out}}[\omega]e^{S^{\mathrm{io}}[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}]}$$

$$= -\frac{i}{2\pi}\langle\varphi^{+}_{\mathrm{out}}[\omega]\rangle = -\frac{i}{2\pi}\langle\hat{b}_{\mathrm{out}}[\omega]\rangle, \tag{2.56}$$

$$\left.\frac{\delta\Lambda_{\mathrm{out}}[\bar{\chi}, \chi^{0}]}{\delta\chi^{0}[\omega]}\right|_{\bar{\chi}=\chi^{0}=0} = -\frac{i}{2\pi} \int \mathcal{D}[\phi, \varphi_{\mathrm{in}}, \varphi_{\mathrm{out}}]\bar{\varphi}_{\mathrm{in}}[\omega]e^{S^{\mathrm{io}}[\phi,\varphi_{\mathrm{in}},\varphi_{\mathrm{out}}]}$$

$$= -\frac{i}{2\pi}\langle\bar{\varphi}_{\mathrm{in}}[\omega]\rangle = -\frac{i}{2\pi}\langle\hat{b}^{y}_{\mathrm{out}}[\omega]\rangle, \tag{2.57}$$

and use these results to rewrite the exponent of the generating functional from Eq. (2.55),

$$S[\bar{\chi}, \chi^{0}] = -\frac{1}{2\pi} \int_{1}^{1} d\omega \Big(\bar{\chi}[\omega]\langle\hat{b}_{\mathrm{out}}[\omega]\rangle + \chi^{0}[\omega]\overline{\langle\hat{b}_{\mathrm{out}}[\omega]\rangle} - in_{B}\bar{\chi}[\omega]\chi^{0}[\omega]\Big). \tag{2.58}$$

These moments are,

$$\langle\hat{b}_{\mathrm{out}}[\omega]\rangle = 2\pi i \cdot \left.\frac{\delta\Lambda_{\mathrm{out}}[\bar{\chi}, \chi^{0}]}{\delta\bar{\chi}[\omega]}\right|_{\bar{\chi}=\chi^{0}=0} = f[\omega]\big(1 - i\kappa G^{R}[\omega]\big), \tag{2.59}$$

$$\langle\hat{b}^{y}_{\mathrm{out}}[\omega]\rangle = 2\pi i \cdot \left.\frac{\delta\Lambda_{\mathrm{out}}[\bar{\chi}, \chi^{0}]}{\delta\chi^{0}[\omega]}\right|_{\bar{\chi}=\chi^{0}=0} = \bar{f}[\omega]\big(1 + i\kappa G^{A}[\omega]\big). \tag{2.60}$$

We again evaluate the $g^{(1)}$–function,

$$\langle \hat{b}^{y}_{\rm out}[\omega_1]\, \hat{b}_{\rm out}[\omega_2]\rangle = (2\pi i)^2 \cdot \frac{\delta^2 \Lambda_{\rm out}[\bar{\chi}, \chi^{0}]}{\delta \chi^{0}[\omega_1]\delta\bar{\chi}[\omega_2]}\bigg|_{\bar{\chi}=\chi^{0}=0} = \overline{\langle \hat{b}_{\rm out}[\omega_1]\rangle}\langle \hat{b}_{\rm out}[\omega_2]\rangle + 2\pi n_B \delta(\omega_1 - \omega_2), \tag{2.61}$$

and the $g^{(2)}$–function,

$$\begin{aligned}
&\langle \hat{b}^{y}_{\rm out}[\omega_1]\, \hat{b}^{y}_{\rm out}[\omega_2]\, \hat{b}_{\rm out}[\omega_3]\, \hat{b}_{\rm out}[\omega_4]\rangle = \\
&(2\pi i)^4 \frac{\delta^4 \Lambda_{\rm out}[\bar{\chi}, \chi^{0}]}{\delta \chi^{0}[\omega_1]\delta \chi^{0}[\omega_2]\delta\bar{\chi}[\omega_3]\delta\bar{\chi}[\omega_4]}\bigg|_{\bar{\chi}=\chi^{0}=0} \\
&= \overline{\langle \hat{b}_{\rm out}[\omega_1]\rangle\langle \hat{b}_{\rm out}[\omega_2]\rangle}\langle \hat{b}_{\rm out}[\omega_3]\rangle\langle \hat{b}_{\rm out}[\omega_4]\rangle \\
&+ (2\pi)^2 n_B \Big(\delta(\omega_1 - \omega_2)\langle \hat{b}_{\rm out}[\omega_4]\rangle\overline{\langle \hat{b}_{\rm out}[\omega_2]\rangle} + \delta(\omega_1 - \omega_4)\langle \hat{b}_{\rm out}[\omega_3]\rangle\overline{\langle \hat{b}_{\rm out}[\omega_2]\rangle} \\
&\qquad + \delta(\omega_3 - \omega_2)\langle \hat{b}_{\rm out}[\omega_4]\rangle\overline{\langle \hat{b}_{\rm out}[\omega_1]\rangle} + \delta(\omega_4 - \omega_2)\langle \hat{b}_{\rm out}[\omega_3]\rangle\overline{\langle \hat{b}_{\rm out}[\omega_1]\rangle}\Big) \\
&+ (2\pi)^4 n_B(\delta(\omega_4 - \omega_2)\delta(\omega_3 - \omega_1) + \delta(\omega_3 - \omega_2)\delta(\omega_4 - \omega_1)).
\end{aligned} \tag{2.62}$$

In the limit of no coupling, $\kappa \to 0$, we get the following expression,

$$\begin{aligned}
&g^{(2)}_{\rm out}[\omega_1, \omega_2, \ldots, \omega_6] \xrightarrow{\kappa\ !\ 0} \big(\bar{f}[\omega_1]\bar{f}[\omega_2]f[\omega_3]f[\omega_4] \\
&+ 2\pi n_B \big(\delta(\omega_4 - \omega_1)f[\omega_3]\bar{f}[\omega_2] + \delta(\omega_3 - \omega_1)f[\omega_4]\bar{f}[\omega_2] \\
&\qquad + \delta(\omega_2 - \omega_4)f[\omega_3]\bar{f}[\omega_1] + \delta(\omega_3 - \omega_2)f[\omega_4]\bar{f}[\omega_1]\big) \\
&+ 4\pi^2 n_B^2 \left(\delta(\omega_2 - \omega_4)\delta(\omega_3 - \omega_1) + \delta(\omega_3 - \omega_2)\delta(\omega_4 - \omega_1)\right)\big) / \big(\bar{f}[\omega_5]f[\omega_6] + 2\pi n_B \delta(\omega_6 - \omega_5)\big).
\end{aligned} \tag{2.63}$$

In the limit of zero temperature, $n_B \to 0$, the expression again factorizes,

$$g^{(2)}_{\rm out}[\omega_1, \omega_2, \ldots, \omega_6] \xrightarrow{n_B\ !\ 0} \frac{\overline{\langle \hat{b}_{\rm out}[\omega_1]\rangle\langle \hat{b}_{\rm out}[\omega_2]\rangle}\langle \hat{b}_{\rm out}[\omega_3]\rangle\langle \hat{b}_{\rm out}[\omega_4]\rangle}{\left(\overline{\langle \hat{b}_{\rm out}[\omega_5]\rangle}\langle \hat{b}_{\rm out}[\omega_6]\rangle\right)^2}. \tag{2.64}$$

2.2.2 Coherent Input

So far our treatment has been independent of the input encoded by the function f. In the last two sections of this chapter we will consider specific input states and evaluate the moments of the output field for those. We start by considering a coherent input state described through the following function as introduced in discrete fashion in Eq. (1.80),

$$f(t) = Ae^{\ \ i\omega_L t}, \qquad\qquad f[\omega] = 2\pi A\delta(\omega - \omega_L). \tag{2.65}$$

We find the reflection coefficient already derived in Sec. 2.1 for the frequency domain,

$$\langle \hat{b}_{\rm out}(t)\rangle = \langle \hat{b}_{\rm in}(t)\rangle \cdot \frac{2(\omega_L - \omega_S) - i\kappa}{2(\omega_L - \omega_S) + i\kappa} = \langle \hat{b}_{\rm in}(t)\rangle \cdot R[\omega_L], \tag{2.66}$$

and that for this specific signal, the output flux exactly matches the input flux, also in accordance with our previous results from Sec. 2.1,

$$\langle \hat{b}^{y}_{\rm in}(t)\, \hat{b}_{\rm in}(t)\rangle = \langle \hat{b}^{y}_{\rm out}(t)\, \hat{b}_{\rm out}(t)\rangle = |A|^2 + n_B\delta(0). \tag{2.67}$$

The Dirac delta distribution appearing here is an artefact of the Markov approximation. Treating the input as white noise results in the same noise being present in the output, these divergences will vanish however for a detector with a finite bandwidth. We can further evaluate the $g^{(2)}$–function which, as expected of a coherent signal, reduces to 1 in the limit of zero temperature,

$$\langle \hat{b}_{\mathrm{in}}^{\dagger}(t)\,\hat{b}_{\mathrm{in}}^{\dagger}(t+\tau)\,\hat{b}_{\mathrm{in}}(t+\tau)\,\hat{b}_{\mathrm{in}}(t)\rangle = \langle \hat{b}_{\mathrm{out}}^{\dagger}(t)\,\hat{b}_{\mathrm{out}}^{\dagger}(t+\tau)\,\hat{b}_{\mathrm{out}}(t+\tau)\,\hat{b}_{\mathrm{out}}(t)\rangle = |A|^4, \tag{2.68}$$

$$g_{\mathrm{in}}^{(2)}(t,\tau) = g_{\mathrm{out}}^{(2)}(\tau) = 1. \tag{2.69}$$

On resonance, $\omega_L = \omega_S$, we further find that the reflection coefficent is equal to -1 and hence we retrieve $\langle \hat{b}_{\mathrm{in}}(t)\rangle = -\langle \hat{b}_{\mathrm{out}}(t)\rangle$ as derived in Eq. (2.12).

2.2.3 Gaussian Input

We now turn to the input being a Gaussian pulse centered around the frequency ω_P in frequency space and around the time μ in the time domain described by the following function,

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-i\omega_P t} e^{-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}}, \qquad f[\omega] = e^{i\mu(\omega-\omega_P)} e^{-\frac{(\omega-\omega_P)^2\sigma^2}{2}}. \tag{2.70}$$

We find the following expression for the average output field,

$$\langle \hat{b}_{\mathrm{out}}(t)\rangle = \langle \hat{b}_{\mathrm{in}}(t)\rangle \cdot \left(1 - \kappa\sigma\sqrt{\frac{\pi}{2}} e^{\frac{\left(2(\mu-t)+\sigma^2(\kappa+2i(\omega_S-\omega_P))\right)^2}{8\sigma^2}} \mathrm{erf}\left(\frac{2(\mu-t)\sigma^2(\kappa+2i(\omega_S-\omega_P))}{\sqrt{8}\sigma}\right)\right), \tag{2.71}$$

which in the resonant case, i.e. $\omega_P = \omega_S$ meaning the Gaussian Pulse being centered around the system frequency, simplifies to,

$$\langle \hat{b}_{\mathrm{out}}(t)\rangle = \langle \hat{b}_{\mathrm{in}}(t)\rangle \cdot \left(1 - \kappa\sigma\sqrt{\frac{\pi}{2}} e^{\frac{\left(2(\mu-t)+\sigma^2\kappa\right)^2}{8\sigma^2}} \mathrm{erf}\left(\frac{2(\mu-t)\sigma^2\kappa}{\sqrt{8}\sigma}\right)\right). \tag{2.72}$$

We can further evaluate the output flux in the resonant case,

$$\begin{aligned}\langle \hat{b}_{\mathrm{out}}^{\dagger}(t)\,\hat{b}_{\mathrm{out}}(t)\rangle &= \langle \hat{b}_{\mathrm{in}}^{\dagger}(t)\,\hat{b}_{\mathrm{in}}(t)\rangle \\ &\quad + \frac{\kappa}{4\sqrt{\pi}\sigma} e^{\frac{\kappa^2\sigma^2}{8}+\frac{\kappa\mu}{2}-\frac{t^2+\mu^2}{2\sigma^2}-t\kappa} \mathrm{erf}\left(\frac{t(\mu-t)+\kappa\sigma^2}{\sqrt{8}\sigma}\right) \\ &\quad \left(e^{\frac{4t^2+\left(2\mu+\kappa\sigma^2\right)^2}{8\sigma^2}} \sqrt{\pi}\kappa\sigma \mathrm{erf}\left(\frac{2(\mu-t)+\kappa\sigma^2}{\sqrt{8}\sigma}\right)\right. \\ &\quad \left. -\sqrt{8} e^{\frac{t\kappa}{2}+\frac{t\mu}{\sigma^2}}\right).\end{aligned} \tag{2.73}$$

In this limit the output flux factorizes into the average of the output field and its hermitian conjugate, $\langle \hat{b}_{\mathrm{out}}^{\dagger}(t)\,\hat{b}_{\mathrm{out}}(t)\rangle = |\langle \hat{b}_{\mathrm{out}}(t)\rangle|^2$, and hence vanishes at times t^* where the answer of the cavity negatively matches the input,

$$f(t^*) = i\kappa \int dt'\, f(t') G^R(t^* - t'). \tag{2.74}$$

See Fig. 2.2 for a specific example where the output flux shows destructive interference with the input signal. The interplay between light entering the cavity and light exiting the cavity leading to this effect here is a general feature of input–output theory with time dependent drives and can lead to non-trivial results.

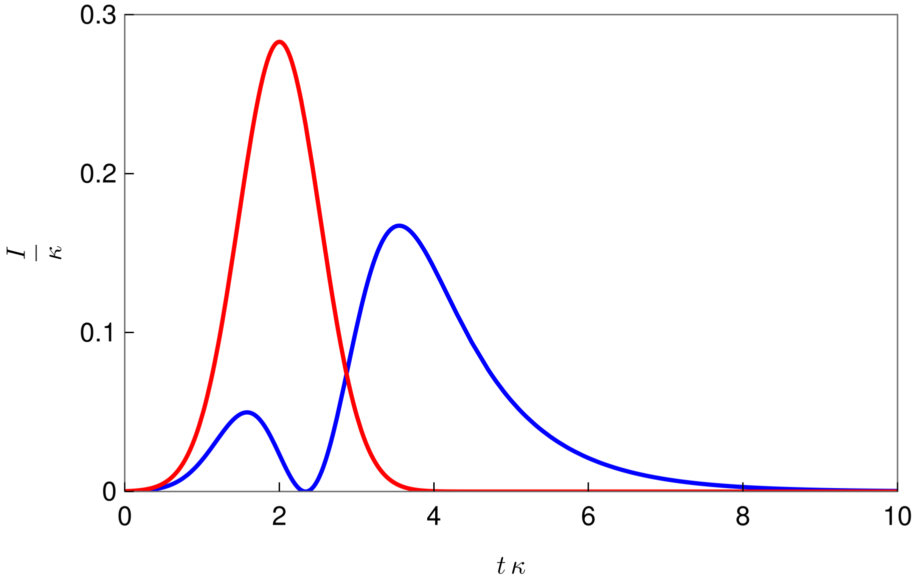

Figure 2.2: Input (red) and output (blue) flux in units of the coupling constant κ for a Gaussian pulse impinging on a cavity with a single mode. The input and output fluxes specify the number of photons arriving per unit time at time t and are thus denoted by an intensity I. At t from Eq. (2.74) destructive interference happens due to the impinging light pulse negatively matching the answer of the cavity.

Chapter 3

Kerr Oscillator

In this chapter we turn to a system exhibiting a non-linearity in its Hamiltonian which will let us showcase how perturbation theory comes naturally into the Keldysh formalism for input–output theory. We will consider the Kerr oscillator, which is a well-studied system exhibiting non–linearities where analytical solutions exist [2, 11].

The Kerr oscillator is described by the following Hamiltonian,

$$\hat{H} = \omega_S \hat{a}^{y}\hat{a} + K\hat{a}^{y}\hat{a}^{y}\hat{a}\hat{a}. \tag{3.1}$$

Obviously, the zeroth–order perturbation in the Kerr parameter K will yield the results already derived in the previous chapter for the damped harmonic oscillator (DHO). Plugging this Hamiltonian into our system–specific part of the action yields the following,

$$\begin{aligned} S_S[\phi] = \sum_{j=1}^{N\ 1} \delta t \left[i\bar{\phi}_j^{+} \frac{\phi_j^{+} - \phi_{j\ 1}^{+}}{\delta t} - \omega_S \bar{\phi}_j^{+}\phi_{j\ 1}^{+} \right] - \sum_{j=1}^{N\ 1} \delta t \left[i\bar{\phi}_j \ \frac{\phi_j \ - \phi_{j\ 1}}{\delta t} - \omega_S \bar{\phi}_{j\ 1}\phi_j \right] \\ + i\bar{\phi}_0\phi_0 - i \ln \rho_S(\bar{\phi}_0, \phi_{2N\ 1}) + \sum_{j=1}^{N\ 1} \delta t K \Big(\big(\bar{\phi}_{j\ 1}\phi_j \big)^2 - \big(\bar{\phi}_j^{+}\phi_{j\ 1}^{+}\big)^2 \Big). \end{aligned} \tag{3.2}$$

In the continuum limit this expression becomes,

$$\begin{aligned} S_S[\phi] &= \int_{t_0}^{t_N} dt[\bar{\phi}^{+}(t)(i\partial_t - \omega_S)\phi^{+}(t) - \bar{\phi} \ (t)(i\partial_t - \omega_S)\phi \ (t) && (3.3) \\ &\qquad + K\Big(\big(\bar{\phi} \ (t)\phi \ (t)\big)^2 - \big(\bar{\phi}^{+}(t)\phi^{+}(t)\big)^2\Big)] \\ &= \int_{t_0}^{t_N} dt[\bar{\phi}^{\mathrm{q}}(t)(i\partial_t - \omega_S)\phi^{\mathrm{cl}}(t) + \bar{\phi}^{\mathrm{cl}}(t)(i\partial_t - \omega_S)\phi^{\mathrm{q}}(t) && (3.4) \\ &\qquad - K\big(\bar{\phi}^{\mathrm{cl}}\bar{\phi}^{\mathrm{q}}\phi^{\mathrm{cl}}\phi^{\mathrm{cl}} + \bar{\phi}^{\mathrm{cl}}\bar{\phi}^{\mathrm{q}}\phi^{\mathrm{q}}\phi^{\mathrm{q}} + \text{ h.c.}\big)]. \end{aligned}$$

Where we again introduced the Keldysh rotation in the last step. The complete action before integrating out the system modes therefore takes the following form,

$$\begin{aligned}
S[\phi,\chi,\chi^0] = &\Bigg[\int_{t_0}^{t_N} dt^0 \int_{t_0}^{t_N} dt \\
&\begin{pmatrix}\bar\phi^{\mathrm{cl}}(t)\\ \bar\phi^{\mathrm{q}}(t)\end{pmatrix}^T \begin{pmatrix} 0 & \big(-\frac{i\kappa}{2} + (i\partial_t - \omega_S)\big)\delta(t-t^0) \\ \big(\frac{i\kappa}{2} + (i\partial_t - \omega_S)\big)\delta(t-t^0) & i\kappa(2n_B+1)\delta(t-t^0)\end{pmatrix}\begin{pmatrix}\phi^{\mathrm{cl}}(t^0)\\ \phi^{\mathrm{q}}(t^0)\end{pmatrix}\Bigg] \\
&+ \int_{t_0}^{t_N} dt \begin{pmatrix} -\chi(t)\sqrt{\frac{\kappa}{2}} \\ -i\bar f(t)\sqrt{2\kappa} - \chi(t)\sqrt{\frac{\kappa}{2}}(2n_B+1)\end{pmatrix}^T \begin{pmatrix}\phi^{\mathrm{cl}}(t)\\ \phi^{\mathrm{q}}(t)\end{pmatrix} \\
&+ \begin{pmatrix} -\chi^0(t)\sqrt{\frac{\kappa}{2}} \\ i f(t)\sqrt{2\kappa} + \chi^0(t)\sqrt{\frac{\kappa}{2}}(2n_B+1)\end{pmatrix}^T \begin{pmatrix}\bar\phi^{\mathrm{cl}}(t)\\ \bar\phi^{\mathrm{q}}(t)\end{pmatrix} \\
&- \chi(t)f(t) - \chi^0(t)\bar f(t) + i n_B \chi(t)\chi^0(t) \\
&- K\big(\bar\phi^{\mathrm{cl}}\bar\phi^{\mathrm{q}}\phi^{\mathrm{cl}}\phi^{\mathrm{cl}} + \bar\phi^{\mathrm{cl}}\bar\phi^{\mathrm{q}}\phi^{\mathrm{q}}\phi^{\mathrm{q}} + \text{ h.c.}\big) \\
= &S_{\mathrm{DHO}}[\phi,\chi,\chi^0] + K\int_{t_0}^{t_N} dt\Big(\big(\bar\phi^{\ }(t)\phi^{\ }(t)\big)^2 - \big(\bar\phi^{+}(t)\phi^{+}(t)\big)^2\Big) \\
= &S_{\mathrm{DHO}}[\phi,\chi,\chi^0] + S_{\mathrm{int}}[\phi]. \qquad (3.5)
\end{aligned}$$

In the last step we wrote the action as a sum of the damped harmonic oscillator action and the additional part stemming from the Kerr term in the Hamiltonian which we call the interacting (int) part borrowing from standard quantum field theory.

3.1 Generating Functional

Now we turn to evaluating the generating functional which we will do perturbatively by expanding the exponential containing the interaction in the Kerr parameter K to first order,

$$\begin{aligned}
\Lambda_{\mathrm{out}}[\chi,\chi^0] = &\int \mathcal{D}[\phi] e^{iS_{\mathrm{DHO}}[\phi,\chi,\chi^0]} e^{iS_{\mathrm{int}}[\phi]} = \int \mathcal{D}[\phi] e^{iS_{\mathrm{DHO}}[\phi,\chi,\chi^0]}\big(1 - iS_{\mathrm{int}}[\phi] + \mathcal{O}(K^2)\big) \\
= &\Lambda_{\mathrm{out}}[\chi,\chi^0]_{\mathrm{DHO}} \qquad (3.6) \\
&- iK\Lambda_{\mathrm{out}}[\chi,\chi^0]_{\mathrm{DHO}} \int dt \Big(\langle\big(\bar\phi^{\ }(t)\phi^{\ }(t)\big)^2\rangle[\chi,\chi^0] - \langle\big(\bar\phi^{+}(t)\phi^{+}(t)\big)^2\rangle[\chi,\chi^0]\Big) + \mathcal{O}(K^2).
\end{aligned}$$

The first term here is the generating functional already known from the damped harmonic oscillator and the second term consists of averages over the expressions in the interaction part. These are however not actual expectation values in that they still contain the source fields χ and χ^0. We use this notation here to simply mean the normalized average values before setting the source fields to zero,

$$\langle f(\phi^+,\phi^{\ })\rangle[\chi,\chi^0] = (\Lambda_{\mathrm{out}}[\chi,\chi^0]_{\mathrm{DHO}})^{\ 1}\int \mathcal{D}[\phi] f(\phi^+,\phi^{\ }) e^{iS_{\mathrm{DHO}}[\phi,\chi,\chi^0]}. \qquad (3.7)$$

We now turn to calculating these expectation values in the expansion. In order to do this we introduce new source fields for classical and quantum field and their hermitian conjugates,

$\phi^{\rm cl}, \phi^{\rm q}, \bar\phi^{\rm cl}, \bar\phi^{\rm q}$, in the action of the damped harmonic oscillator,

$$
\begin{aligned}
&S[\phi, \chi, \dots] = \\
&\left[\int_{t_0}^{t_N} dt' \int_{t_0}^{t_N} dt \begin{pmatrix} \bar\phi^{\rm cl}(t) \\ \bar\phi^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} 0 & \left(-\frac{i\kappa}{2} + (i\partial_t - \omega_S)\right)\delta(t-t') \\ \left(\frac{i\kappa}{2} + (i\partial_t - \omega_S)\right)\delta(t-t') & i\kappa(2n_B+1)\delta(t-t') \end{pmatrix} \begin{pmatrix} \phi^{\rm cl}(t') \\ \phi^{\rm q}(t') \end{pmatrix}\right] \\
&+ \int_{t_0}^{t_N} dt \begin{pmatrix} -\chi(t)\sqrt{\frac{\kappa}{2}} - \chi^{\rm cl}(t) \\ -i\bar f(t)\sqrt{2\kappa} - \chi(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) - \chi^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} \phi^{\rm cl}(t) \\ \phi^{\rm q}(t) \end{pmatrix} \\
&+ \begin{pmatrix} -\chi'(t)\sqrt{\frac{\kappa}{2}} - \chi'^{\rm cl}(t) \\ if(t)\sqrt{2\kappa} + \chi'(t)\sqrt{\frac{\kappa}{2}}(2n_B+1) - \chi'^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} \bar\phi^{\rm cl}(t) \\ \bar\phi^{\rm q}(t) \end{pmatrix} \\
&- \chi(t)f(t) - \chi'(t)\bar f(t) + i\chi(t)\chi'(t). \qquad (3.8)
\end{aligned}
$$

After integrating out the system fields we arrive at the functional Γ,

$$\Gamma[\chi, \chi', \chi^{\rm cl}, \chi^{\rm q}, \chi'^{\rm cl}, \chi'^{\rm q}] = \exp\left\{iS[\chi, \chi', \chi^{\rm cl}, \chi^{\rm q}, \chi'^{\rm cl}, \chi'^{\rm q}]\right\}, \qquad (3.9)$$

from which we get the expectation values of the different fields through functional derivatives of the following form,

$$
\begin{aligned}
\langle\bar\phi^{\rm cl}(t)\rangle[\chi, \chi'] &= \left(\Lambda_{\rm out}[\chi, \chi']_{\rm DHO}\right)^{-1} i \left.\frac{\delta\Gamma}{\delta\chi'^{\rm cl}(t)}\right|_{\chi^{\rm cl}=\chi^{\rm q}=\chi'^{\rm cl}=\chi'^{\rm q}=0} \\
&= \left(\Lambda_{\rm out}[\chi, \chi']_{\rm DHO}\right)^{-1} \int \mathcal{D}[\phi]\bar\phi^{\rm cl}(t) e^{iS_{\rm DHO}[\phi,\chi,\chi']}. \qquad (3.10)
\end{aligned}
$$

The action can now again be split up into the action of the damped harmonic oscillator and an additional part,

$$
\begin{aligned}
S = S_{\rm DHO}[\chi, \chi'] - \int dt &\left[\begin{pmatrix} \chi^{\rm cl}(t) \\ \chi^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} \langle\phi^{\rm cl}(t)\rangle \\ \langle\phi^{\rm q}(t)\rangle \end{pmatrix} + \begin{pmatrix} \chi'^{\rm cl}(t) \\ \chi'^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} \langle\bar\phi^{\rm cl}(t)\rangle \\ \langle\bar\phi^{\rm q}(t)\rangle \end{pmatrix}\right. \qquad (3.11) \\
&\left. + \int dt' \begin{pmatrix} \chi^{\rm cl}(t) \\ \chi^{\rm q}(t) \end{pmatrix}^T \begin{pmatrix} G^K(t-t') & G^R(t-t') \\ G^A(t-t') & 0 \end{pmatrix} \begin{pmatrix} \chi'^{\rm cl}(t') \\ \chi'^{\rm q}(t') \end{pmatrix}\right] \\
= S_{\rm DHO}&[\chi, \chi'] \\
- \int dt &\left[\vec\chi(t)^T \begin{pmatrix} \langle\phi^{\rm cl}(t)\rangle \\ \langle\phi^{\rm q}(t)\rangle \end{pmatrix} + \vec\chi'(t)^T \begin{pmatrix} \langle\bar\phi^{\rm cl}(t)\rangle \\ \langle\bar\phi^{\rm q}(t)\rangle \end{pmatrix} + \int dt'\, \vec\chi(t)^T \underline{G}(t-t')\vec\chi'(t')\right],
\end{aligned}
$$

with,

$$\langle\phi^{\rm cl}(t)\rangle[\chi'] = \int dt' \left[\sqrt{\frac{\kappa}{2}}\left(G^K(t-t')\chi'(t') - G^R(t-t')\chi'(t')F\right) - i\sqrt{2\kappa}G^R(t-t')f(t')\right], \quad (3.12)$$

$$\langle\bar\phi^{\rm cl}(t)\rangle[\chi] = \int dt' \left[\sqrt{\frac{\kappa}{2}}\left(G^K(t'-t)\chi(t') + G^A(t'-t)\chi(t')F\right) + i\sqrt{2\kappa}G^A(t'-t)\bar f(t')\right], \quad (3.13)$$

$$\langle\phi^{\rm q}(t)\rangle[\chi'] = \int dt' \sqrt{\frac{\kappa}{2}} G^A(t-t')\chi'(t'), \qquad (3.14)$$

$$\langle\bar\phi^{\rm q}(t)\rangle[\chi] = \int dt' \sqrt{\frac{\kappa}{2}} G^R(t'-t)\chi(t'). \qquad (3.15)$$

From here we can calculate $\langle \phi \ \ (t)\rangle$ by using the definition of the Keldysh rotation,

$$\begin{aligned}
\langle \phi^{+}(t)\rangle[\chi^{0}] &= \frac{1}{\sqrt{2}}\big(\langle \phi^{\mathrm{cl}}(t)\rangle[\chi^{0}] + \langle \phi^{\mathrm{q}}(t)\rangle[\chi^{0}]\big) \\
&= -\int dt^{0}\sqrt{\kappa}\left[\frac{F-1}{2}G^{A}(t-t^{0})\chi^{0}(t^{0}) + iG^{R}(t-t^{0})f(t^{0})\right], \qquad (3.16)\\
\langle \phi \ \ (t)\rangle[\chi^{0}] &= \frac{1}{\sqrt{2}}\big(\langle \phi^{\mathrm{cl}}(t)\rangle[\chi^{0}] - \langle \phi^{\mathrm{q}}(t)\rangle[\chi^{0}]\big) \\
&= -\int dt^{0}\sqrt{\kappa}\left[\frac{F+1}{2}G^{A}(t-t^{0})\chi^{0}(t^{0}) + iG^{R}(t-t^{0})f(t^{0})\right], \qquad (3.17)\\
\langle \bar{\phi}^{+}(t)\rangle[\chi] &= \frac{1}{\sqrt{2}}\big(\langle \bar{\phi}^{\mathrm{cl}}(t)\rangle[\chi] + \langle \bar{\phi}^{\mathrm{q}}(t)\rangle[\chi]\big) \\
&= \int dt^{0}\sqrt{\kappa}\left[\frac{F+1}{2}G^{R}(t^{0}-t)\chi(t^{0}) + iG^{A}(t^{0}-t)\bar{f}(t^{0})\right], \qquad (3.18)\\
\langle \bar{\phi} \ \ (t)\rangle[\chi] &= \frac{1}{\sqrt{2}}\big(\langle \bar{\phi}^{\mathrm{cl}}(t)\rangle[\chi] - \langle \bar{\phi}^{\mathrm{q}}(t)\rangle[\chi]\big) \\
&= \int dt^{0}\sqrt{\kappa}\left[\frac{F-1}{2}G^{R}(t^{0}-t)\chi(t^{0}) + iG^{A}(t^{0}-t)\bar{f}(t^{0})\right]. \qquad (3.19)
\end{aligned}$$

Where we made the following and similar simplifications,

$$\begin{aligned}
&G^{K}(t-t^{0}) + G^{A}(t-t^{0}) - G^{R}(t-t^{0})F \\
&= \underbrace{G^{K}(t-t^{0}) + FG^{A}(t-t^{0}) - G^{R}(t-t^{0})F}_{=0} + G^{A}(t-t^{0}) - G^{A}(t-t^{0})F = 2(F-1)G^{A}(t-t^{0}).
\end{aligned} \qquad (3.20)$$

We are eventually interested in the terms $\langle\big(\bar{\phi} \ \ \phi \ \ \big)^{2}\rangle[\chi,\chi^{0}]$ that arise after expanding the interaction term containing the Kerr parameter. To this end we introduce the shifted fields,

$$\delta\phi \ \ = \phi \ \ - \langle\phi \ \ \rangle[\chi,\chi^{0}], \qquad (3.21)$$

which fulfill,

$$\begin{aligned}
\langle\delta\phi \ \ \rangle &= (\Lambda_{\mathrm{out}}[\chi,\chi^{0}]) \ \ ^{1}\left(\int \mathcal{D}[\chi]\phi \ \ e^{iS_{\mathrm{DHO}}[\phi,\chi,\chi^{0}]} - \Lambda_{\mathrm{out}}[\chi,\chi^{0}]_{\mathrm{DHO}}\langle\phi \ \ \rangle[\chi,\chi^{0}]\right) \\
&= 0. \qquad (3.22)
\end{aligned}$$

We firstly evaluate $\langle\big(\delta\bar{\phi} \ \ \delta\phi \ \ \big)^{2}\rangle$ by using Wicks Theorem,

$$\langle\big(\delta\bar{\phi} \ \ \delta\phi \ \ \big)^{2}\rangle = 2\langle\delta\bar{\phi} \ \ \delta\phi \ \ \rangle^{2}. \qquad (3.23)$$

The expressions for $\langle\delta\bar{\phi} \ \ \phi \ \ \rangle$ can be taken from [7] since this is now purely quadratic in the system fields and the source fields only enter in the linear terms of the action,

$$\begin{aligned}
\langle\delta\bar{\phi}^{+}(t^{0})\delta\phi^{+}(t)\rangle &= \theta(t-t^{0})iG^{>}(t,t^{0}) + \theta(t^{0}-t)iG^{<}(t,t^{0}), \qquad (3.24)\\
\langle\delta\bar{\phi} \ \ (t^{0})\delta\phi \ \ (t)\rangle &= \theta(t^{0}-t)iG^{>}(t,t^{0}) + \theta(t-t^{0})iG^{<}(t,t^{0}). \qquad (3.25)
\end{aligned}$$

In order to evaluate $\langle\delta\bar{\phi} \ \ (t)\delta\phi \ \ (t)\rangle$ we have to revert back to the discrete notation to determine the time ordering. Because even though in the continuum notation it seems as if these two fields are evaluated at the same time, looking back at the discrete notation in Eq. (3.2) however, we see that one is actually evaluated before the other. In keeping with the continuum notation from Eqs. (3.25)&(3.24) we find that $t^{0} > t$ on the forward time branch and $t^{0} < t$ on the backwards time branch. Therefore,

$$\langle\delta\bar{\phi}^{+}(t^{0})\delta\phi^{+}(t)\rangle = iG^{<}(t,t^{0}) \rightarrow n_B, \tag{3.26}$$

$$\langle\delta\bar{\phi}\ \ (t^{0})\delta\phi\ \ (t)\rangle = iG^{<}(t,t^{0}) \rightarrow n_B. \tag{3.27}$$

On the other hand we can write out $\langle\big(\delta\bar{\phi}\ \ \delta\phi\ \ \big)^2\rangle$ by plugging in the definition of $\delta\phi$ and thereby get an expression for $\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^{0}]$,

$$\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^{0}] = \langle\big(\big(\delta\bar{\phi}\ \ + \langle\bar{\phi}\ \ \rangle[\chi,\chi^{0}]\big)\big(\delta\phi\ \ + \langle\phi\ \ \rangle[\chi,\chi^{0}]\big)\big)^2\rangle. \tag{3.28}$$

Using Wicks Theorem and the fact that $\langle\delta\phi\ \ \rangle = \langle\delta\phi\ \ \delta\phi\ \ \rangle = 0$, we get,

$$\begin{aligned}\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle = {}& 2\langle\delta\phi\ \ \delta\phi\ \ \rangle^2 + \langle\delta\bar{\phi}\ \ \delta\phi\ \ \rangle\langle\bar{\phi}\ \ \rangle\langle\phi\ \ \rangle + \langle\delta\bar{\phi}\ \ \delta\phi\ \ \rangle\langle\phi\ \ \rangle\langle\bar{\phi}\ \ \rangle \\ & + \langle\delta\bar{\phi}\ \ \delta\phi\ \ \rangle\langle\bar{\phi}\ \ \rangle\langle\phi\ \ \rangle + \langle\bar{\phi}\ \ \rangle^2\langle\phi\ \ \rangle^2. \end{aligned} \tag{3.29}$$

Here we suppressed the dependence on χ and χ^{0} for notational ease. With this we find,

$$\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^{0}] = 2n_B^2 + 3n_B\langle\bar{\phi}\ \ \rangle[\chi,\chi^{0}]\langle\phi\ \ \rangle[\chi,\chi^{0}] + \langle\bar{\phi}\ \ \rangle^2[\chi,\chi^{0}]\langle\phi\ \ \rangle^2[\chi,\chi^{0}], \tag{3.30}$$

and can now express the generating functional for the Kerr oscillator to first order in K through the Green functions,

$$\begin{aligned}\Lambda_{\text{out}}[\chi,\chi^{0}] = {}&\Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}} \\ & - iK\Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}}\int dt\,\big[3n_B\big(\langle\bar{\phi}\ \ \rangle[\chi,\chi^{0}]\langle\phi\ \ \rangle[\chi,\chi^{0}] - \langle\bar{\phi}^{+}\rangle[\chi,\chi^{0}]\langle\phi^{+}\rangle[\chi,\chi^{0}]\big)\big) \\ & + \langle\bar{\phi}\ \ \rangle^2[\chi,\chi^{0}]\langle\phi\ \ \rangle^2[\chi,\chi^{0}] - \langle\bar{\phi}^{+}\rangle^2[\chi,\chi^{0}]\langle\phi^{+}\rangle^2[\chi,\chi^{0}]]] + \mathcal{O}(K^2). \end{aligned} \tag{3.31}$$

From the generating functional, we can derive the cumulant generating functional by taking the logarithm. For this we first rewrite the generating functional as follows,

$$\begin{aligned}\Lambda_{\text{out}}[\chi,\chi^{0}] &= \Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}} \\ &\quad - iK\Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}}\int dt\Big(\langle\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2\rangle[\chi,\chi^{0}] - \langle\big(\bar{\phi}^{+}(t)\phi^{+}(t)\big)^2\rangle[\chi,\chi^{0}]\Big) + \mathcal{O}(K^2) \\ &= \Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}}\Big(1 - iK\int dt\Big(\langle\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2\rangle[\chi,\chi^{0}] - \langle\big(\bar{\phi}^{+}(t)\phi^{+}(t)\big)^2\rangle[\chi,\chi^{0}]\Big)\Big) + \mathcal{O}(K^2) \\ &= \Lambda_{\text{out}}[\chi,\chi^{0}]_{\text{DHO}}e^{\ \ iK\int dt\big(h\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2 i[\chi,\chi^{0}]\ \ h\big(\bar{\phi}^{+}(t)\phi^{+}(t)\big)^2 i[\chi,\chi^{0}]\big)} + \mathcal{O}(K^2). \end{aligned} \tag{3.32}$$

Now taking the logarithm is straighforward and results in the cumulant generating functional of the output field,

$$\begin{aligned}\mathcal{S}[\chi,\chi^{0}]_{\text{Kerr}} = {}&\mathcal{S}[\chi,\chi^{0}]_{\text{DHO}} - iK\int dt\,\big[3n_B\big(\langle\bar{\phi}\ \ \rangle[\chi]\langle\phi\ \ \rangle[\chi^{0}] - \langle\bar{\phi}^{+}\rangle[\chi]\langle\phi^{+}\rangle[\chi^{0}]\big)\big) \\ & + \langle\bar{\phi}\ \ \rangle^2[\chi]\langle\phi\ \ \rangle^2[\chi^{0}] - \langle\bar{\phi}^{+}\rangle^2[\chi]\langle\phi^{+}\rangle^2[\chi^{0}]]] + \mathcal{O}(K^2). \end{aligned} \tag{3.33}$$

From either the generating functional or the cumulant generating functional we can now derive the moments or cumulants of the output field to first order in the Kerr parameter K.

3.2 Statistics of the Output Field

We start by computing the expectation value of the output field for the Kerr oscillator to first order in K through the generating functional from the previous subsection,

$$\langle\hat{b}_{\text{out}}(u)\rangle_{\text{Kerr}} = \langle\hat{b}_{\text{out}}(u)\rangle_{\text{DHO}}$$
$$+ iK \int dt G^R(u-t) \bigg(3n_B\kappa \int dt_1 G^R(t-t_1) f(t_1)$$
$$+2\kappa^2 \iiint dt_3 dt_4 dt_5 G^R(t-t_3) f(t_3) G^R(t-t_4) f(t_4) G^A(t_5-t) \bar{f}(t_5) \bigg) + \mathcal{O}(K^2). \tag{3.34}$$

For details on the calculation see B.2.1. For zero temperature (i.e. $n_B = 0$) this reduces to,

$$\begin{aligned}\langle\hat{b}_{\text{out}}(u)\rangle_{\text{Kerr}} =& \langle\hat{b}_{\text{out}}(u)\rangle_{\text{DHO}} \\ &+2iK\kappa^2 \iiiint dt dt_3 dt_4 dt_5 G^R(u-t) G^R(t-t_3) f(t_3) \\ &\qquad G^R(t-t_4) f(t_4) G^A(t_5-t) \bar{f}(t_5) + \mathcal{O}(K^2),\end{aligned} \tag{3.35}$$

and for $K \to 0$ we retrieve the expected average value of the damped harmonic oscillator. From the cumulant generating functional we can compute the first moment again, this time however with much less work,

$$\langle\hat{b}_{\text{out}}(u)\rangle_{\text{Kerr}} = i \frac{\delta\mathcal{S}[\chi,\chi^0]_{\text{Kerr}}}{\delta\chi(u)}\bigg|_{\chi=\chi^0=0} = \langle\hat{b}_{\text{out}}(u)\rangle_{\text{DHO}}$$
$$+ iK \int dt G^R(u-t) \bigg(3n_B\kappa \int dt_1 G^R(t-t_1) f(t_1)$$
$$+2\kappa^2 \iiint dt_3 dt_4 dt_5 G^R(t-t_3) f(t_3) G^R(t-t_4) f(t_4) G^A(t_5-t) \bar{f}(t_5) \bigg) + \mathcal{O}(K^2). \tag{3.36}$$

Similarly we can compute the expectation value of the hermitian conjugate of the output field,

$$\langle\hat{b}^{y}_{\text{out}}(u)\rangle_{\text{Kerr}} = i \frac{\delta\mathcal{S}[\chi,\chi^0]_{\text{Kerr}}}{\delta\chi^0(u)}\bigg|_{\chi=\chi^0=0} = \langle\hat{b}^{y}_{\text{out}}(u)\rangle_{\text{DHO}}$$
$$- iK \int dt G^A(t-u) \bigg(3n_B\kappa \int dt_1 G^A(t_1-t) \bar{f}(t_1)$$
$$+2\kappa^2 \iiint dt_2 dt_3 dt_4 G^A(t_2-t) \bar{f}(t_2) G^A(t_3-t) \bar{f}(t_3) G^R(t_4-t) f(t_4) \bigg) + \mathcal{O}(K^2), \tag{3.37}$$

which is exactly the complex conjugate of $\langle\hat{b}_{\text{out}}(u)\rangle_{\text{Kerr}}$. We turn to second order cumulants,

$$\langle\langle\hat{b}^{y}_{\text{out}}(u)\,\hat{b}_{\text{out}}(w)\rangle\rangle = (i)^2 \left.\frac{\delta\mathcal{S}[\chi,\chi^0]}{\delta\chi^0(u)\delta\chi(w)}\right|_{\chi=\chi^0=0} = \langle\hat{b}^{y}_{\text{out}}(u)\,\hat{b}_{\text{out}}(w)\rangle - \langle\hat{b}^{y}_{\text{out}}(u)\rangle\langle\hat{b}_{\text{out}}(w)\rangle. \tag{3.38}$$

For the Kerr oscillator we find that these cumulants match those from the damped harmonic oscillator to first order in K,

$$\langle\langle\hat{b}^{y}_{\text{out}}(u)\,\hat{b}_{\text{out}}(w)\rangle\rangle_{\text{Kerr}} = \langle\langle\hat{b}^{y}_{\text{out}}(u)\,\hat{b}_{\text{out}}(w)\rangle\rangle_{\text{DHO}} + \mathcal{O}(K^2), \tag{3.39}$$

which leads to the output flux taking the following form,

$$\langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\,\hat{b}_{\mathrm{out}}(w)\rangle_{\mathrm{Kerr}} = \langle\langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\,\hat{b}_{\mathrm{out}}(w)\rangle\rangle_{\mathrm{DHO}} + \langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{Kerr}}\langle \hat{b}_{\mathrm{out}}(w)\rangle_{\mathrm{Kerr}} + \mathcal{O}(K^2) \tag{3.40}$$

$$= n_B\delta(u-w) + \langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{Kerr}}\langle \hat{b}_{\mathrm{out}}(w)\rangle_{\mathrm{Kerr}} + \mathcal{O}(K^2). \tag{3.41}$$

From the expression of the cumulant generating functional in Eq. (3.33) it is evident that all cumulants of higher order than four must vanish. The remaining non–zero cumulants are thus the following,

$$\langle\langle \hat{b}^{\dagger}_{\mathrm{out}}(u_1)\,\hat{b}^{\dagger}_{\mathrm{out}}(u_2)\,\hat{b}_{\mathrm{out}}(u_3)\rangle\rangle_{\mathrm{Kerr}} =$$
$$iK\kappa^2\big(2+6F^2\big)\iint dt\,dt'\,G^R(u_1-t)G^A(t-u_2)G^A(t-u_3)G^A(t'-t)\bar{f}(t') + \mathcal{O}(K^2), \tag{3.42}$$

$$\langle\langle \hat{b}^{\dagger}_{\mathrm{out}}(u_1)\,\hat{b}^{\dagger}_{\mathrm{out}}(u_2)\,\hat{b}_{\mathrm{out}}(u_3)\,\hat{b}_{\mathrm{out}}(u_4)\rangle\rangle_{\mathrm{Kerr}} = \mathcal{O}(K^2). \tag{3.43}$$

Contrary to the case of the damped harmonic oscillator in Chap. 2, the third cummulant does not vanish for the Kerr oscillator. With this we have completely characterized the statistics of the output field for the Kerr oscillator to first order in the Kerr parameter K.

3.3 Coherent Input

So far our treatment has been independent of the input encoded by the function f. In the final section of this chapter we will consider a coherent input state described through the following function,

$$f(t) = Ae^{-i\omega_L t}, \qquad f[\omega] = 2\pi A\delta(\omega-\omega_L). \tag{3.44}$$

For the amplitude A being equal to one, the average output field takes the following form,

$$\langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{Kerr}} = \langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{DHO}} + \frac{4ie^{-iu\omega_L}K\kappa\big(8\kappa+3n_B\big(\kappa^2+4(\omega_L-\omega_S)^2\big)\big)}{(i\kappa+2(\omega_L-\omega_S))((\kappa+2i\omega_L)^2+4\omega_S^2)} + \mathcal{O}(K^2). \tag{3.45}$$

The averages values of the damped harmonic oscillator (DHO) can be found in Sec.2.2.2. On resonance and in the limit of zero temperature this expression simplifies to,

$$\langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{Kerr}} = \langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\rangle_{\mathrm{DHO}} + 32e^{-iu\omega_L}\frac{K}{i\kappa^2-4\kappa\omega_S} + \mathcal{O}(K^2). \tag{3.46}$$

At zero temperature and with the detuning $\Delta = \omega_S - \omega_L$ the output flux takes the following form,

$$\langle \hat{b}^{\dagger}_{\mathrm{out}}(u)\,\hat{b}_{\mathrm{out}}(w)\rangle_{\mathrm{Kerr}} = 1 + 256\frac{K\kappa^2\omega_S}{(4\Delta^2+\kappa^2)^2\Big(\kappa^2+4(\Delta-2\omega_S)^2\Big)} + \mathcal{O}(K^2). \tag{3.47}$$

In both Eq. (3.46) and Eq. (3.47) we see a divergence for the coupling constant κ and the detuning Δ going to zero. This traces back to the fact that $\langle\hat{a}\rangle$ diverges in the limit of zero coupling and resonant driving, as can already be seen in Eq. (2.10) from Sec. 2.1. Hence the number of photons in the cavity goes to infinity and perturbation theory in the Kerr term of the Kerr Hamiltonian breaks down.

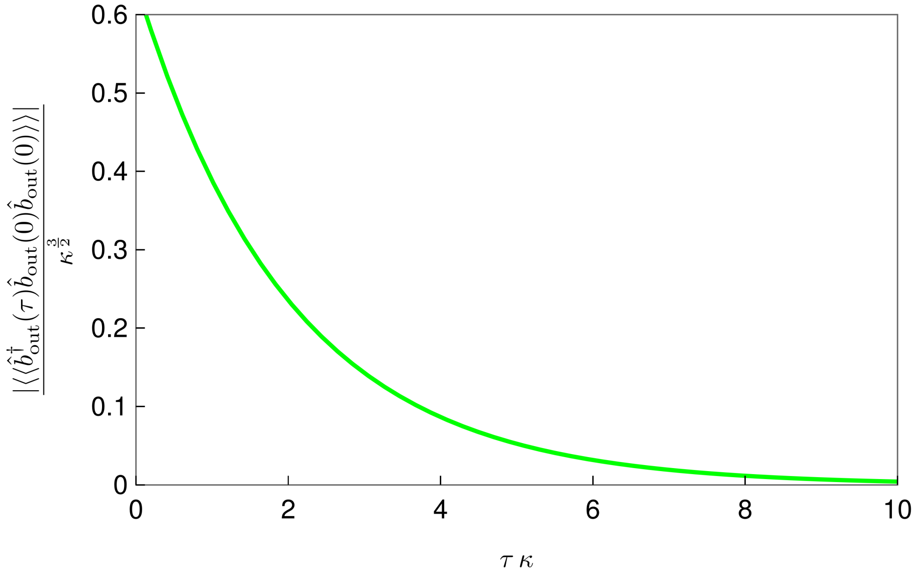

Figure 3.1: Absolute value of the third cumulant for $^{K}/_{\kappa} = 0.06$, a coherent input signal and where we set $u_2 = u_3 = 0$ and varied $u_1 = \tau$ in Eq. (3.42).

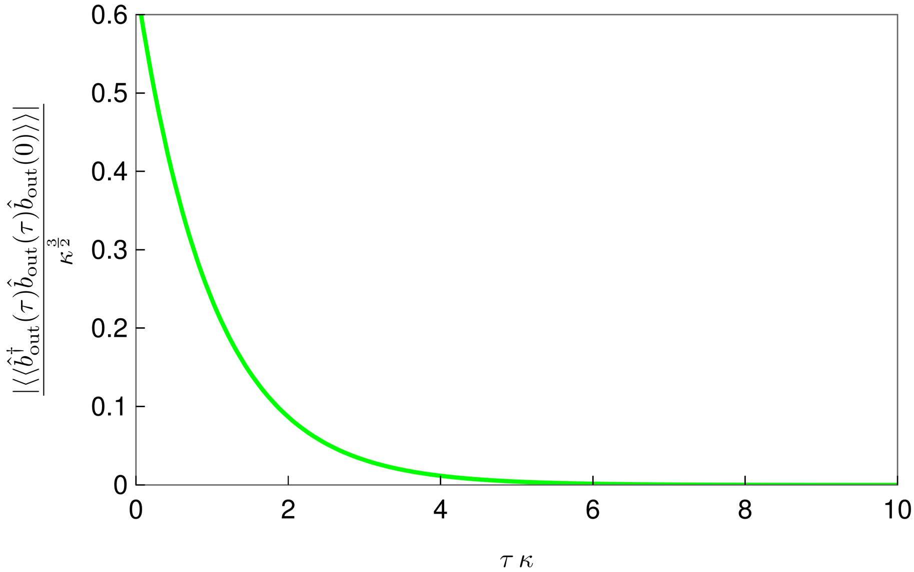

Figure 3.2: Absolute value of the third cumulant for $^{K}/_{\kappa} = 0.06$, a coherent input signal and where we set $u_3 = 0$ and varied $u_1 = u_2 = \tau$ in Eq. (3.42).

Conclusion and Outlook

In this thesis we have explained the standard approach to input–output theory in Sec. 1.2, given an overview over the Keldysh path integral formalism in Sec. 1.3 and detailed the derivation and application of our novel approach to input–output theory through the use of the Keldysh path integral in Sec. 1.3.

The path integral based approach we presented led us to the well–known input–output relation by means of a stationary phase approximation in our path integral, see Sec. 1.4.4, and we defined its most central quantity with the moment generating functional of the output field. From this quantity, once evaluated exactly or perturbatively for a specified system, we showed how one can derive the moments of the output field and evaluate the P-functional.

From this general treatment of the formalism, we moved on to two examples in Chap. 2 and Chap. 3 to demonstrate our approach.

We first showcased our method through the solvable system of a single mode, i.e. a damped harmonic oscillator in our setting. We solved for the statistics of the output field through both the standard approach to input–output theory, Sec. 2.1, as well as our novel path–integral based approach, Sec. 2.2. Both approaches gave the same exact results without the need for perturbation theory. In this treatment we devoted special attention to two input signals, namely a time independent coherent signal, see Sec. 2.2.2, and a time dependent Gaussian pulse, see Sec. 2.2.3. The latter showed interference effects between the input and output flux characteristic for the setup we consider. No new results were found nor expected in this chapter and it served as an introduction and test case for our approach. After this proof of concept, we treated the Kerr oscillator in our novel formalism and derived perturbative results for the statistics of the output field to linear order in the Kerr parameter. This was achieved by calculating the moment generating functional as well as the cumulant generating functional in Sec. 3.1 and subsequent derivation of the non–zero moments and cumulants of the system in Sec. 3.2. Interestingly, we found that contrary to the damped harmonic oscillator, the third cumulant of the output field does not vanish for the Kerr oscillator in linear response theory.

Having presented the path integral approach to input–output theory and after applying it to scenarios where comparison to existing results was possible, the next step will be to apply the formalism to scenarios where it is difficult to obtain results with standard input–output theory. A first step in this direction will be to investigate the parametric oscillator which is known to produce squeezed states of light and then extend that line of inquiry to the Kerr parametric oscillator which has the curious feature of displaying a negative Wigner function in the output field for a positive Wigner function in the intra–cavity field [14].

Appendix A

Appendix A contains additional information, theoretical prerequisites or omitted details in calculations regarding the material presented in the chapter on theoretical formalism, Chap. 1.

A.1 Standard Input–Output Theory

A.1.1 Method of Variation of Parameters

For a first order inhomogeneous linear differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = p(t)y(t) + f(t),$$

the solution, found by *variation of parameters*, is given by,

$$y(t) = v(t)e^{P(t)} + Ae^{P(t)},$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = e^{-P(t)}f(t),$$

and $P(t)$ is the anti-derivative of $p(t)$.

A.1.2 Retrieving System Modes from Input and Output

We make use of the following two identities,

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t'), \tag{A.1}$$

$$\int_{t_0}^{t} d\tau f(\tau)\delta(\tau - t) = \frac{1}{2}f(t). \tag{A.2}$$

We prove the first relation, Eq. (1.13), the second one follows from analogy.

$$\begin{aligned}
&\frac{\sqrt{\kappa}}{2\pi i g_k}\int_{\omega_k-\delta\omega}^{\omega_k+\delta\omega} d\omega \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t_0)}\, \hat{b}_{\mathrm{in}}(\mathrm{t}) \\
&= \frac{1}{2\pi g_k}\sum_l \int_{\omega_k-\delta\omega}^{\omega_k+\delta\omega} d\omega \int_{-\infty}^{\infty} dt\, g_l e^{it_0(\omega_l-\omega)} e^{-it(\omega-\omega_l)}\hat{b}_l(t_0).
\end{aligned}$$

Integration over t yields a delta distribution,

$$\dots = \frac{1}{g_k} \sum_l \int_{\omega_k - \delta\omega}^{\omega_k + \delta\omega} d\omega \, g_l e^{it_0(\omega_l - \omega)} \delta(\omega - \omega_l) \hat{b}_l(t_0),$$

and as $\delta\omega$ is chosen s.t. $\delta\omega \le \omega_l - \omega_{l-1} \ \forall l$, we find,

$$\dots = \frac{1}{g_k} g_k \hat{b}_k(t_0) = \hat{b}_k(t_0).$$

A.2 Keldysh Input–Output Action

A.2.1 Bosonic Gaussian Integral

We cite the formula for computing bosonic Gaussian integrals from [1],

$$\int \mathcal{D}[\phi] e^{-\frac{1}{2} \begin{pmatrix} \bar{\phi} \\ \phi \end{pmatrix}^T \Gamma \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix} + \begin{pmatrix} J_1 \\ J_2 \end{pmatrix} \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}} = \frac{1}{\sqrt{\det\{\Gamma\}}} e^{\frac{1}{2} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}^T \Gamma^{-1} \begin{pmatrix} J_2 \\ J_1 \end{pmatrix}}. \tag{A.3}$$

A.2.2 Integrating out Bath Modes

What follows is a detailed description of the step in which we integrate out all the bath modes from the system that are neither set at $t_0 = t_{2N-1}$ nor $t_N = t_{N-1}$. The notation here can be confusing, the exponents S_k for $k = 1, 2, 3, 4, 5, 6$ have no imaginary unit factor i contrary to the way actions are usually written in quantum field theory. We are faced with the following integral,

$$Z = \int \left(\prod_{j=0}^{2N-1} d[\phi_j] e^{iS_S[\phi]} \right) \left(\prod_k d[\varphi_{k,0}] d[\varphi_{k,N-1}] d[\varphi_{k,N}] d[\varphi_{k,2N-1}] e^{S_1} e^{S_2} \right) \underbrace{\left(\prod_{j=1}^{N-2} \prod_k d[\varphi_{k,j}] e^{S_3} e^{S_4} \right)}_{:=\Delta} \underbrace{\left(\prod_{j=N+2}^{2N-2} \prod_k d[\varphi_{k,j}] e^{S_5} e^{S_6} \right)}_{:=\square}, \tag{A.4}$$

with,

$$
\begin{aligned}
S_1 &= -\ln \rho_K(\bar{\varphi}_0, \varphi_{2N\;\;1}) \\
&\quad - \sum_k \left[\varphi_{k,0}\bar{\varphi}_{k,0} - \bar{\varphi}_{k,N\;\;1}\varphi_{k,N\;\;1} - \bar{\varphi}_{k,N}\varphi_{k,N} + \bar{\varphi}_{k,N}\varphi_{k,N\;\;1} - \bar{\varphi}_{k,2N\;\;1}\varphi_{k,2N\;\;1}\right], &\text{(A.5)} \\
S_2 &= i\delta t\big(g_k\bar{\phi}_1\varphi_{k,0} + g_k\bar{\varphi}_{k,N\;\;1}\phi_{N\;\;2} - g_k\bar{\phi}_{N+1}\varphi_{k,N} - g_k\bar{\varphi}_{k,2N\;\;1}\phi_{2N\;\;1}\big), &\text{(A.6)} \\
S_3 &= -\bar{\varphi}_{k,1}\varphi_{k,1} - \sum_{j=2}^{N\;\;2}\left[\bar{\varphi}_{k,j}\varphi_{k,j} - \bar{\varphi}_{k,j}\varphi_{k,j\;\;1} + i\delta t\omega_k\bar{\varphi}_{k,j}\varphi_{k,j\;\;1}\right], &\text{(A.7)} \\
S_4 &= \bar{\varphi}_{k,1}\varphi_{k,0} \\
&\quad -i\delta t(\omega_k\bar{\varphi}_{k,1}\varphi_{k,0} + g_k\bar{\varphi}_{k,1}\phi_0 + \sum_{j=2}^{N\;\;2}\left[g_k\bar{\phi}_j\varphi_{k,j\;\;1} + g_k\bar{\varphi}_{k,j}\phi_{j\;\;1}\right] &\text{(A.8)} \\
&\qquad + g_k\bar{\phi}_{N\;\;1}\varphi_{k,N\;\;2} + \omega_k\bar{\varphi}_{k,N\;\;1}\varphi_{k,N\;\;2}) \\
&\quad + \bar{\varphi}_{k,N\;\;1}\varphi_{k,N\;\;2}, \\
S_5 &= -\bar{\varphi}_{k,N+1}\varphi_{k,N+1} - \sum_{j=N+2}^{2N\;\;2}\left[\bar{\varphi}_{k,j}\varphi_{k,j} - \bar{\varphi}_{k,j}\varphi_{k,j\;\;1} - i\delta t\omega_k\bar{\varphi}_{k,j}\varphi_{k,j\;\;1}\right], &\text{(A.9)} \\
S_6 &= \bar{\varphi}_{k,N+1}\varphi_{k,N} + i\delta t(\omega_k\bar{\varphi}_{k,N+1}\varphi_{k,N} + ig_k\bar{\varphi}_{k,N+1}\phi_N \\
&\quad + \sum_{j=N+2}^{2N\;\;2}\left[g_k\bar{\phi}_j\varphi_{k,j\;\;1} + g_k\bar{\varphi}_{k,j}\phi_{j\;\;1}\right] + g_k\bar{\phi}_{2N\;\;1}\varphi_{k,2N\;\;2} + \omega_k\bar{\varphi}_{k,2N\;\;1}\varphi_{k,2N\;\;2}) \\
&\quad + \bar{\varphi}_{k,2N\;\;1}\varphi_{k,2N\;\;2}. &\text{(A.10)}
\end{aligned}
$$

We now evaluate the integral containing the terms in what we denoted with Δ in Eq. (A.4), evaluating the integral containing   follows from analogy. We use the formula for multidimensional Gaussian integrals, Eq. (A.3), to evaluate the integral for a fixed k,

$$
\Gamma = \begin{pmatrix} \Gamma^{(1)} & 0 \\ 0 & \Gamma^{(2)} \end{pmatrix} \in \mathbb{C}^{(2N\;\;4)\;\;(2N\;\;4)}, \tag{A.11}
$$

$$
\Gamma^{(1)} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
h & 1 & 0 & \cdots & \cdots & 0 \\
0 & h & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & & \vdots \\
\vdots & & & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & h & 1
\end{pmatrix}, \tag{A.12}
$$

$$
\Gamma^{(2)} = \left(\Gamma^{(1)}\right)^T \in \mathbb{C}^{(N\;\;2)\;\;(N\;\;2)}. \tag{A.13}
$$

Where we introduced

$$
h = -(1 - i\delta t\omega_k) = -e^{\;\;i\delta t\omega_k} + \mathcal{O}\big(\delta t^2\big). \tag{A.14}
$$

$$J_1 = \left[i\delta t g_k \begin{pmatrix} \bar{\phi}_2 \\ \bar{\phi}_3 \\ \vdots \\ \vdots \\ \bar{\phi}_{N-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ h\bar{\varphi}_{k,N-1} \end{pmatrix} \right] \in \mathbb{C}^{N-2}, \tag{A.15}$$

$$J_2 = \left[i\delta t g_k \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \vdots \\ \phi_{N-3} \end{pmatrix} + \begin{pmatrix} h\varphi_{k,0} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right] \in \mathbb{C}^{N-2}. \tag{A.16}$$

We invert Γ,

$$\Gamma^{-1} = \begin{pmatrix} \left(\Gamma^{(1)}\right)^{-1} & 0 \\ 0 & \left(\Gamma^{(2)}\right)^{-1} \end{pmatrix} \left(\Gamma^{(1)}\right)^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -h & 1 & 0 & \cdots & \cdots & 0 \\ h^2 & -h & 1 & 0 & \cdots & 0 \\ -h^3 & h^2 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ (-h)^{N-3} & \cdots & -h^3 & h^2 & -h & 1 \end{pmatrix}, \tag{A.17}$$

$$\left(\Gamma^{(2)}\right)^{-1} = \left(\left(\Gamma^{(1)}\right)^{-1}\right)^T. \tag{A.18}$$

With Γ^{-1} we can determine the exponent as follows,

$$\begin{pmatrix} J_1 \\ J_2 \end{pmatrix}^T \Gamma^{-1} \begin{pmatrix} J_2 \\ J_1 \end{pmatrix} = J_1^T \left(\Gamma^{(1)}\right)^{-1} J_2 + J_2^T \left(\Gamma^{(2)}\right)^{-1} J_1 = 2 \cdot \left(J_1^T \left(\Gamma^{(1)}\right)^{-1} J_2 \right). \tag{A.19}$$

We turn to evaluating the expression above,

$$\left(\Gamma^{(1)}\right)^{-1} J_2 = \left[i\delta t g_k \begin{pmatrix} \phi_0 \\ \phi_1 - h\phi_0 \\ \phi_2 - h\phi_1 + h^2\phi_0 \\ \vdots \\ \sum_{l=1}^{N-2} \phi_{l-1}(-h)^{N-2-l} \end{pmatrix} - \varphi_{k,0} \begin{pmatrix} -h \\ h^2 \\ -h^3 \\ \vdots \\ (-h)^{N-2}. \end{pmatrix} \right], \tag{A.20}$$

with this we can conclude,

$$\begin{aligned} J_1^T \left(\Gamma^{(1)}\right)^{-1} J_2 &= -\delta t^2 |g_k|^2 \sum_{j=1}^{N-2} \sum_{l=1}^{j} \phi_{l-1} \bar{\phi}_{j+1} (-h)^{j-l} - i\delta t g_k \varphi_{k,0} \sum_{j=2}^{N-1} \bar{\phi}_j (-h)^{j-1} \\ &\quad - i\delta t g_k \bar{\varphi}_{k,N-1} \sum_{j=1}^{N-2} \phi_{j-1} (-h)^{N-1-j} - \bar{\varphi}_{k,N-1} \varphi_{k,0} (-h)^{N-1} \\ &= -\delta t^2 |g_k|^2 \sum_{j=1}^{N-2} \sum_{l=1}^{j} \phi_{l-1} \bar{\phi}_{j+1} e^{-i\omega_k (t_j - t_l)} - i\delta t g_k \varphi_{k,0} \sum_{j=2}^{N-1} \bar{\phi}_j e^{-i\omega_k (t_{j-1} - t_0)} \\ &\quad - i\delta t g_k \bar{\varphi}_{k,N-1} \sum_{j=1}^{N-2} \phi_{j-1} e^{-i\omega_k (t_N - t_0)} - \bar{\varphi}_{k,N-1} \varphi_{k,0} e^{-i\omega_k (t_N - t_0)} + \mathcal{O}\left(\delta t^2\right). \end{aligned} \tag{A.21}$$

Where we used $h = -e^{\;i\delta t\omega_k} + \mathcal{O}\big(\delta t^2\big)$ in the last equality and introduced,

$$t_n = t_0 + \sum_{m=1}^{n} \delta t_n. \tag{A.22}$$

Note that this implies $t_{N\;\;1} = t_N$. With $\det\{\Gamma\} = \det\big\{\Gamma^{(1)}\big\} \cdot \det\big\{\Gamma^{(2)}\big\} = 1$, we can conclude our calculation of Δ,

$$\Delta = e^{J_1^T\left(\Gamma^{(1)}\right)\;\;^1 J_2}, \tag{A.23}$$

and by analogy evaluate .

A.2.3 Input States

Thermal State

With a canonical thermal state, or Gibbs state, for a system with Hamiltonian $\hat{H}$ and inverse temperature β, we mean the following density matrix,

$$\hat{\rho}_G = \frac{1}{Z} e^{\;\;\beta\hat{H}}. \tag{A.24}$$

With Z being the thermodynamic partition function,

$$Z = \mathrm{Tr}\Big\{e^{\;\;\beta\hat{H}}\Big\} = \sum_{n=0}^{1} \langle n| \, e^{\;\;\beta\hat{H}} \, |n\rangle = \sum_{n=0}^{1} e^{\;\;\beta E_n}, \tag{A.25}$$

assuming an equally spaced spectrum of energy eigenvalues we can further evaluate,

$$\sum_{n=0}^{1} e^{\;\;\beta E_n} = \sum_{n=0}^{1} e^{\;\;\beta\omega n} = \frac{1}{1 - e^{\;\;\beta\omega}}. \tag{A.26}$$

Where we employed the formula for a geometric series in the last step, assuming $|e^{\;\;\beta\omega}| \leq 1$, $\forall n$. Another important identity is the matrix element of the thermal state in the basis of coherent states. Let $|\varphi\rangle, |\varphi^0\rangle$ be two coherent states,

$$\langle\varphi| \, e^{\;\;\beta\hat{H}} \, |\varphi^0\rangle = \left(\sum_{n=0}^{1} \frac{\bar{\varphi}^n}{\sqrt{n!}} \langle n|\right) e^{\;\;\beta\hat{H}} \left(\sum_{m=0}^{1} \frac{\varphi^{0m}}{\sqrt{m!}} |m\rangle\right) = \sum_{n=0}^{1} \frac{(\bar{\varphi}\varphi^0)^n}{n!} e^{\;\;\beta E_n}, \tag{A.27}$$

assuming an equally spaced spectrum of energy eigenvalues we can further evaluate,

$$\sum_{n=0}^{1} \frac{(\bar{\varphi}\varphi^0)^n}{n!} e^{\;\;\beta E_n} = \sum_{n=0}^{1} \frac{\left(\bar{\varphi}\varphi^0 e^{\;\;\beta\omega}\right)^n}{n!} = e^{\bar{\varphi}\varphi^0 e^{\;\;\beta\omega}}. \tag{A.28}$$

Product of Coherent States

$$\rho_B(\varphi_{\mathrm{in}}) = \exp\left[\sum_{j=0}^{N\;\;1} \delta t\Big(f_j \bar{\varphi}^{+}_{\mathrm{in},j} + \bar{f}_j \varphi_{\mathrm{in},j} - |f_j|^2\Big)\right]. \tag{A.29}$$

The first two terms are self-explanatory, the third term emerges as follows,

$$\begin{aligned}\sum_{j=0}^{N-1} \delta t|f_j|^2 &= \sum_{j=0}^{N-1} \delta t \bar{f}_j f_j = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \bar{\alpha}_k \alpha_l e^{i(t_j-t_0)(\omega_k-\omega_l)} \\ &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} \bar{\alpha}_k \alpha_l e^{2\pi i \frac{j}{N}(k-l)} = \sum_k |\alpha_k|^2, \end{aligned} \tag{A.30}$$

where we used

$$i(t_j - t_0)(\omega_k - \omega_l) = i\delta t j \delta\omega(k-l) = i\frac{2\pi}{N\cancel{\delta\omega}} j\cancel{\delta\omega}(k-l) = 2\pi i \frac{j}{N}(k-l). \tag{A.31}$$

Kronecker Delta Sum Identity

We recall the following identity,

$$\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(n-m)} = \delta_{n,m}. \tag{A.32}$$

For the case $n = m$ the identity is trivial, for the case $n \neq m$ one can prove it using the formula for the geometric series,

$$\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{k}{N}(n-m)} = \frac{1}{N}\left(\frac{1-1}{1-e^{2\pi i \frac{1}{N}(n-m)}}\right) = 0. \tag{A.33}$$

Baker-Campbell-Hausdorff Formula

For two possibly non-commuting operators X, Y the product of their exponentials is given by,

$$e^X e^Y = e^Z, \tag{A.34}$$

with

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots. \tag{A.35}$$

When X and Y commute with their commutator that implies the following relation,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{A.36}$$

Displacement Operator

The displacement operator is defined as follows [6],

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}, \tag{A.37}$$

and gets its name from the following relation,

$$\hat{D}(\alpha)\left|0\right\rangle = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\bar{\alpha}\hat{a}}\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger}\left|0\right\rangle = e^{-\frac{|\alpha|^2}{2}}\left|\alpha\right\rangle, \tag{A.38}$$

where we used Eq. (A.36), Eq. (1.32) and $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ in the last two steps. We prove a few useful relations,

$$\hat{D}^\dagger(\alpha) = e^{\alpha\hat{a} - \alpha\hat{a}^\dagger} = e^{-\left(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}\right)} = \hat{D}(-\alpha), \tag{A.39}$$

$$\hat{D}(-\alpha)\hat{D}(\beta) = e^{-\alpha\hat{a}^\dagger + \bar{\alpha}\hat{a}} e^{\beta\hat{a}^\dagger - \bar{\beta}\hat{a}} = e^{(\beta-\alpha)\hat{a}^\dagger - \overline{(\beta-\alpha)}\hat{a}} e^{\frac{1}{2}(\bar{\alpha}\beta - \bar{\beta}\alpha)} = \hat{D}(\beta-\alpha) e^{\frac{1}{2}(\bar{\alpha}\beta - \bar{\beta}\alpha)}, \tag{A.40}$$

$$\hat{D}(-\alpha)\left|\beta\right\rangle = e^{\frac{|\beta|^2}{2}} \hat{D}(-\alpha)\hat{D}(\beta)\left|0\right\rangle = e^{\frac{|\beta|^2}{2}} e^{\frac{1}{2}(\bar{\alpha}\beta - \bar{\beta}\alpha)} e^{-\frac{|\beta-\alpha|^2}{2}} \left|\beta-\alpha\right\rangle = e^{\beta\bar{\alpha} - \frac{|\alpha|^2}{2}} \left|\beta-\alpha\right\rangle. \tag{A.41}$$

A.2.4 Stationary Phase Approximation

Functional Derivative

We can define a functional derivative of a functional $F[f]$ as follows,

$$\frac{\delta F}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{F[f(x') + \epsilon\delta(x - x')] - F[f(x')]}{\epsilon}. \tag{A.42}$$

(cited from [10]).

A.2.5 Statistics of the Input Field

The discrete version of the generating functional for the input field is given as follows,

$$\Lambda_{\text{in}}[\chi, \chi'] = e^{-\sum_{j=0}^{N-1} i\chi_j f_j + i\chi'_j \bar{f}_j + \chi_j\chi'_j n_B}, \tag{A.43}$$

from which one can derive expectation values like the following first moment,

$$i\frac{\partial}{\partial\chi_j}\Lambda_{\text{in}}[\chi, \chi']\bigg|_{\chi=\chi'=0} = f_j. \tag{A.44}$$

A.2.6 Statistics of the Output Field

Discrete Version of the Generating Functional

We compute the generating functional for the moments of the output field in the discrete notation. For this we start with the Keldysh input-output action from Eq. (1.65) and add source fields χ, χ' to the fields corresponding to $\hat{b}_{\text{out}}(t_j)$ and $\hat{b}^\dagger_{\text{out}}(t_j)$ on the forward and backward timebranch respectively.

$$\Lambda_{\text{out}}[\chi, \chi'] = \int \mathcal{D}[\phi, \varphi_{\text{in}}, \varphi_{\text{out}}] e^{iS^{\text{io}}[\phi, \varphi_{\text{in}}, \varphi_{\text{out}}] - i\sum_{j=0}^{N-1} \chi_j \varphi^+_{\text{out},j} + \chi'_j \bar{\varphi}_{\text{out},j}}, \tag{A.45}$$

with

$$S^{\text{io}}[\phi, \varphi_{\text{in}}, \varphi_{\text{out}}] = S^{\text{io}}_S[\phi] + S^{\text{io}}_B[\varphi_{\text{in}}, \varphi_{\text{out}}] + S^{\text{io}}_V[\phi, \varphi_{\text{out}}, \varphi_{\text{in}}], \tag{A.46}$$

where

$$S_B^{\text{io}}[\varphi_{\text{in}}, \varphi_{\text{out}}] = i\delta t \sum_{j=0}^{N\ 1} \bar{\varphi}^+_{\text{in},j}\varphi^+_{\text{in},j} - \bar{\varphi}^+_{\text{out},j}\varphi^+_{\text{in},j} + \bar{\varphi}^+_{\text{out},j}\varphi^+_{\text{out},j} \tag{A.47}$$

$$- \bar{\varphi}_{\text{out},j}\varphi^+_{\text{out},j} + \bar{\varphi}_{\text{out},j}\varphi_{\text{out},j} - \bar{\varphi}_{\text{in},j}\varphi_{\text{out},j} + \bar{\varphi}_{\text{in},j}\varphi_{\text{in},j} - i\ln\rho_B(\varphi_{\text{in}}),$$

$$S_V^{\text{io}}[\phi, \varphi_{\text{in}}, \varphi_{\text{out}}] = i\delta t\sqrt{\kappa} \sum_{j=1}^{N\ 1} \bar{\phi}^+_j \varphi^+_{\text{in},j\ 1} - \phi^+_{j\ 1}\bar{\varphi}^+_{\text{out},j} - \bar{\phi}_{j\ 1}\varphi_{\text{out},j} + \bar{\varphi}_{\text{in},j}\phi_j\,. \tag{A.48}$$

We start by integrating out the output-bath modes using the following form of the multidimensional Gaussian integral,

$$\int \mathcal{D}[z] e^{\ \bar{z}^T\Gamma z + J_1^T z + J_2^T \bar{z}} = \frac{1}{\det\{\Gamma\}}^2 e^{J_1\Gamma\ ^1 J_2}. \tag{A.49}$$

In our case $z = \begin{pmatrix}\varphi^+_{\text{out},0} & \cdots & \varphi^+_{\text{out},N\ 1} & \varphi_{\text{out},0} & \cdots & \varphi_{\text{out},N\ 1}\end{pmatrix}^T$. We read off from the expressions in Eq. (A.46),

$$\Gamma = \delta t \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ -1 & & & & 1 & & & \\ & -1 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & -1 & & & & 1 \end{pmatrix} \tag{A.50}$$

inverting this matrix yields,

$$\Gamma^{\ 1} = \frac{1}{\delta t} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 1 \end{pmatrix}. \tag{A.51}$$

We can further read of J_1, J_2,

$$J_1 = -i \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{N\ 1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \delta t \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\varphi}_{\text{in},0} \\ \bar{\varphi}_{\text{in},1} \\ \vdots \\ \bar{\varphi}_{\text{in},N\ 1} \end{pmatrix} + \delta t\sqrt{\kappa} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \bar{\phi}_0 \\ \vdots \\ \bar{\phi}_{N\ 2} \end{pmatrix}, \tag{A.52}$$

$$J_2 = -i \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi_0^{0} \\ \chi_1^{0} \\ \vdots \\ \chi_{N\ \ 1}^{0} \end{pmatrix} + \delta t \begin{pmatrix} \varphi_{\text{in},0}^{+} \\ \varphi_{\text{in},1}^{+} \\ \vdots \\ \varphi_{\text{in},N\ \ 1}^{+} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \delta t \sqrt{\kappa} \begin{pmatrix} 0 \\ \phi_0^{+} \\ \vdots \\ \phi_{N\ \ 2}^{+} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{A.53}$$

Now we calculate the exponent. First we evaluate $\Gamma^{\ \ 1} J_2$,

$$\Gamma^{\ \ 1} J_2 = -\frac{i}{\delta t} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi_0^{0} \\ \chi_1^{0} \\ \vdots \\ \chi_{N\ \ 1}^{0} \end{pmatrix} + \begin{pmatrix} \varphi_{\text{in},0}^{+} \\ \varphi_{\text{in},1}^{+} \\ \vdots \\ \varphi_{\text{in},N\ \ 1}^{+} \\ \varphi_{\text{in},0}^{+} \\ \varphi_{\text{in},1}^{+} \\ \vdots \\ \varphi_{\text{in},N\ \ 1}^{+} \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ \phi_0^{+} \\ \vdots \\ \phi_{N\ \ 2}^{+} \\ 0 \\ \phi_0^{+} \\ \vdots \\ \phi_{N\ \ 2}^{+} \end{pmatrix}, \tag{A.54}$$

and now the full term,

$$J_1 \Gamma^{\ \ 1} J_2 = \left[-i \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{N\ \ 1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \delta t \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\varphi}_{\text{in},0} \\ \bar{\varphi}_{\text{in},1} \\ \vdots \\ \bar{\varphi}_{\text{in},N\ \ 1} \end{pmatrix} + \delta t \sqrt{\kappa} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \bar{\phi}_0 \\ \vdots \\ \bar{\phi}_{N\ \ 2} \end{pmatrix} \right] \tag{A.55}$$

$$\left[-\frac{i}{\delta t} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi_0^{0} \\ \chi_1^{0} \\ \vdots \\ \chi_{N\ \ 1}^{0} \end{pmatrix} + \begin{pmatrix} \varphi_{\text{in},0}^{+} \\ \varphi_{\text{in},1}^{+} \\ \vdots \\ \varphi_{\text{in},N\ \ 1}^{+} \\ \varphi_{\text{in},0}^{+} \\ \varphi_{\text{in},1}^{+} \\ \vdots \\ \varphi_{\text{in},N\ \ 1}^{+} \end{pmatrix} + \kappa \begin{pmatrix} 0 \\ \phi_0^{+} \\ \vdots \\ \phi_{N\ \ 2}^{+} \\ 0 \\ \phi_0^{+} \\ \vdots \\ \phi_{N\ \ 2}^{+} \end{pmatrix} \right]$$

$$\begin{aligned} = & -i \sum_{j=0}^{N\ \ 1} \left[\chi_j \varphi_{\text{in},j}^{+} + \chi_j^{0} \bar{\varphi}_{\text{in},j} \right] - i\sqrt{\kappa} \sum_{j=1}^{N\ \ 1} \left[\chi_j \phi_{j\ \ 1}^{+} + \chi_j^{0} \bar{\phi}_{j\ \ 1} \right] + \delta t \sum_{j=0}^{N\ \ 1} \varphi_{\text{in},j}^{+} \bar{\varphi}_{\text{in},j} \\ & + \delta t \sqrt{\kappa} \sum_{j=1}^{N\ \ 1} \left[\bar{\varphi}_{\text{in},j} \phi_{j\ \ 1}^{+} + \varphi_{\text{in},j}^{+} \bar{\phi}_{j\ \ 1} \right] + \delta t \kappa \sum_{j=1}^{N\ \ 1} \bar{\phi}_{j\ \ 1} \phi_{j\ \ 1}^{+}. \end{aligned} \tag{A.56}$$

This result agrees with the continuous version found previously. After integrating out the output–modes the generating functional takes the following form,

$$\Lambda_{\text{out}}[\chi, \chi'] = \int \mathcal{D}[\phi, \varphi_{\text{in}}] e^{iS[\phi, \varphi_{\text{in}}]}, \tag{A.57}$$

with

$$\begin{aligned} S[\phi, \varphi_{\text{in}}] = S_S^{\text{io}}[\phi] - i\delta t\kappa \sum_{j=1}^{N-1} \bar{\phi}_{j-1} \phi_{j-1}^{+} - \sqrt{\kappa} \sum_{j=1}^{N-1} \left[\chi_j \phi_{j-1}^{+} + \chi'_j \bar{\phi}_{j-1}\right] \\ + i\delta t \sum_{j=0}^{N-1} \left[\bar{\varphi}_{\text{in},j}^{+} \varphi_{\text{in},j}^{+} + \bar{\varphi}_{\text{in},j} \varphi_{\text{in},j} - \varphi_{\text{in},j}^{+} \bar{\varphi}_{\text{in},j}\right] - i \ln \rho_B(\varphi_{\text{in}}) \\ - \sum_{j=0}^{N-1} \left[\chi_j \varphi_{\text{in},j}^{+} + \chi'_j \bar{\varphi}_{\text{in},j}\right] \\ + i\delta t\sqrt{\kappa} \sum_{j=1}^{N-1} \left[\bar{\phi}_j^{+} \varphi_{\text{in},j-1}^{+} + \bar{\varphi}_{\text{in},j-1} \phi_j - \bar{\varphi}_{\text{in},j} \phi_{j-1}^{+} - \varphi_{\text{in},j}^{+} \bar{\phi}_{j-1}\right]. \end{aligned} \tag{A.58}$$

Now we integrate out the input–modes with the same procedure as above.

$$\Gamma = \delta t \begin{pmatrix} 1 & & & & -e^{-\beta\Omega} & & & \\ & 1 & & & & -e^{-\beta\Omega} & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & -e^{-\beta\Omega} \\ -1 & & & & 1 & & & \\ & -1 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & -1 & & & & 1 \end{pmatrix} \tag{A.59}$$

inverting this matrix yields,

$$\begin{aligned} \Gamma^{-1} &= \frac{1}{\delta t} \frac{1}{e^{\beta\Omega} - 1} \begin{pmatrix} e^{\beta\Omega} & & & & 1 & & & \\ & e^{\beta\Omega} & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & e^{\beta\Omega} & & & & 1 \\ e^{\beta\Omega} & & & & e^{\beta\Omega} & & & \\ & e^{\beta\Omega} & & & & e^{\beta\Omega} & & \\ & & \ddots & & & & \ddots & \\ & & & e^{\beta\Omega} & & & & e^{\beta\Omega} \end{pmatrix} \\ &= \frac{1}{\delta t} \begin{pmatrix} 1+n_B & & & & n_B & & & \\ & 1+n_B & & & & n_B & & \\ & & \ddots & & & & \ddots & \\ & & & 1+n_B & & & & n_B \\ 1+n_B & & & & 1+n_B & & & \\ & 1+n_B & & & & 1+n_B & & \\ & & \ddots & & & & \ddots & \\ & & & 1+n_B & & & & 1+n_B \end{pmatrix} \end{aligned} \tag{A.60}$$

We can further read of J_1, J_2

$$J_1 = -i\begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{N\ 1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \delta t\sqrt{\kappa}\begin{pmatrix} \bar{\phi}_1^+ \\ \vdots \\ \bar{\phi}_{N\ 1}^+ \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \delta t\sqrt{\kappa}\begin{pmatrix} 0 \\ \bar{\phi}_0 \\ \vdots \\ \bar{\phi}_{N\ 2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{\delta t}{1+n_B}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{f}_0 \\ \bar{f}_1 \\ \vdots \\ \bar{f}_{N\ 1} \end{pmatrix}, \tag{A.61}$$

$$J_2 = -i\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi_0^{0} \\ \chi_1^{0} \\ \vdots \\ \chi_{N\ 1}^{0} \end{pmatrix} - \delta t\sqrt{\kappa}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \phi_1 \\ \vdots \\ \phi_{N\ 2} \\ 0 \end{pmatrix} + \delta t\sqrt{\kappa}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \phi_0^+ \\ \vdots \\ \phi_{N\ 1}^+ \end{pmatrix} + \frac{\delta t}{1+n_B}\begin{pmatrix} f_0 \\ f_2 \\ \vdots \\ f_{N\ 1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{A.62}$$

Now we calculate the exponent. First we evaluate $\Gamma\ ^1 J_2$,

$$\Gamma\ ^1 J_2 = -\frac{i}{\delta t}\begin{pmatrix} n_B\chi_0^{0} \\ n_B\chi_1^{0} \\ \vdots \\ n_B\chi_{N\ 1}^{0} \\ (1+n_B)\chi_0^{0} \\ (1+n_B)\chi_1^{0} \\ \vdots \\ (1+n_B)\chi_{N\ 1}^{0} \end{pmatrix} - \sqrt{\kappa}\begin{pmatrix} n_B\phi_1 \\ \vdots \\ n_B\phi_{N\ 1} \\ 0 \\ (1+n_B)\phi_1 \\ \vdots \\ (1+n_B)\phi_{N\ 1} \\ 0 \end{pmatrix} + \sqrt{\kappa}\begin{pmatrix} 0 \\ n_B\phi_0^+ \\ \vdots \\ n_B\phi_{N\ 2}^+ \\ 0 \\ (1+n_B)\phi_0^+ \\ \vdots \\ (1+n_B)\phi_{N\ 2}^+ \end{pmatrix} + \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N\ 1} \\ f_0 \\ f_1 \\ \vdots \\ f_{N\ 1} \end{pmatrix}. \tag{A.63}$$

and now the full term,

$$J_1\Gamma^{\ 1}J_2 = \left[-i\begin{pmatrix}\chi_0\\ \chi_1\\ \vdots\\ \chi_{N\ 1}\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} - \delta t\sqrt{\kappa}\begin{pmatrix}\bar{\phi}_1^{+}\\ \vdots\\ \bar{\phi}_{N\ 1}^{+}\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} + \delta t\sqrt{\kappa}\begin{pmatrix}0\\ \bar{\phi}_0\\ \vdots\\ \bar{\phi}_{N\ 2}\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} + \frac{\delta t}{1+n_B}\begin{pmatrix}0\\ 0\\ \vdots\\ 0\\ \bar{f}_0\\ \bar{f}_1\\ \vdots\\ \bar{f}_{N\ 1}\end{pmatrix}\right]$$

$$\left[-\frac{i}{\delta t}\begin{pmatrix}n_B\chi_0^{0}\\ n_B\chi_1^{0}\\ \vdots\\ n_B\chi_{N\ 1}^{0}\\ (1+n_B)\chi_0^{0}\\ (1+n_B)\chi_1^{0}\\ \vdots\\ (1+n_B)\chi_{N\ 1}^{0}\end{pmatrix} - \sqrt{\kappa}\begin{pmatrix}n_B\phi_1\\ \vdots\\ n_B\phi_{N\ 1}\\ 0\\ (1+n_B)\phi_1\\ \vdots\\ (1+n_B)\phi_{N\ 1}\\ 0\end{pmatrix} + \sqrt{\kappa}\begin{pmatrix}0\\ n_B\phi_0^{+}\\ \vdots\\ n_B\phi_{N\ 2}^{+}\\ 0\\ (1+n_B)\phi_0^{+}\\ \vdots\\ (1+n_B)\phi_{N\ 2}^{+}\end{pmatrix} + \begin{pmatrix}f_0\\ f_1\\ \vdots\\ f_{N\ 1}\\ f_0\\ f_1\\ \vdots\\ f_{N\ 1}\end{pmatrix}\right].$$

$$\begin{aligned}J_1\Gamma^{\ 1}J_2 = &-\frac{n_B}{\delta t}\sum_{j=0}^{N\ 1}\chi_j\chi_j^{0}\\ &+i\sqrt{\kappa}n_B\sum_{j=1}^{N\ 1}\left[\chi_{j\ 1}\phi_j - \chi_j\phi_{j\ 1}^{+} + \bar{\phi}_j^{+}\chi_{j\ 1}^{0} - \bar{\phi}_{j\ 1}\chi_j^{0}\right]\\ &-i\sum_{j=0}^{N\ 1}\left[\chi_j f_j + \bar{f}_j\chi_j^{0}\right]\\ &+\delta t\kappa n_B\sum_{j=1}^{N\ 1}\left[\bar{\phi}_j^{+}\phi_j - \bar{\phi}_j^{+}\phi_{j\ 1}^{+} - \bar{\phi}_{j\ 1}\phi_j + \bar{\phi}_{j\ 1}\phi_{j\ 1}^{+}\right]\\ &+\delta t\sqrt{\kappa}\sum_{j=1}^{N\ 1}\left[\phi_{j\ 1}^{+}\bar{f}_j - \phi_j\bar{f}_{j\ 1}\right] + \frac{\delta t}{1+n_B}\sum_{j=0}^{N\ 1}\bar{f}_j f_j.\end{aligned} \tag{A.64}$$

After integrating out the input–modes the generating functional takes the following form,

$$\Lambda_{\text{out}}[\chi,\chi^{0}] = \int \mathcal{D}[\phi]e^{iS[\phi]}, \tag{A.65}$$

with

$$\boxed{\begin{aligned} S[\phi,\chi,\chi^{0}] =& S_S^{\mathrm{io}}[\phi] - i\delta t\kappa \sum_{j=1}^{N\;1} \bar{\phi}_{j\;1}\phi^{+}_{j\;1} - \sqrt{\kappa} \sum_{j=1}^{N\;1} \left[\chi_j\phi^{+}_{j\;1} + \chi^{0}_j\bar{\phi}_{j\;1}\right] + i\frac{n_B}{\delta t}\sum_{j=0}^{N\;1}\chi_j\chi^{0}_j + \\ & \sqrt{\kappa} n_B \sum_{j=1}^{N\;1}\left[\chi_{j\;1}\phi_{j\;} - \chi_j\phi^{+}_{j\;1} + \bar{\phi}^{+}_j\chi^{0}_{j\;1} - \bar{\phi}_{j\;1}\chi^{0}_j\right] - \sum_{j=0}^{N\;1}\left[\chi_j f_j + \bar{f}_j\chi^{0}_j\right] \\ & - i\delta t\kappa n_B \sum_{j=1}^{N\;1}\left[\bar{\phi}^{+}_j\phi_{j\;} - \bar{\phi}^{+}_j\phi^{+}_{j\;1} - \bar{\phi}_{j\;1}\phi_{j\;} + \bar{\phi}_{j\;1}\phi^{+}_{j\;1}\right] \\ & - i\delta t\sqrt{\kappa}\sum_{j=1}^{N\;1}\left[\phi^{+}_{j\;1}\bar{f}_j - \phi_{j\;}\bar{f}_{j\;1}\right]. \end{aligned}}$$

(A.66)

Appendix B

Appendix B contains additional information, theoretical prerequisites or omitted details in calculations regarding the material presented in the chapters on the damped harmonic oscillator and the Kerr oscillator, Chap. 2 and Chap. 3.

B.1 Damped Harmonic Oscillator

Fourier Transform

We are using the following convention of the Fourier transform,

$$f[\omega] = \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}, \qquad\qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega f[\omega] e^{-i\omega t} \tag{B.1}$$

This is the convention used in [3].

B.2 Kerr Oscillator

B.2.1 Output Field

Calculation of First Moment

$$\begin{aligned} \langle \hat{b}_{\mathrm{out}}(u) \rangle_{\mathrm{Kerr}} &= i \frac{\delta \Lambda_{\mathrm{Kerr}}[\chi, \chi^0]}{\delta \chi(u)} \bigg|_{\chi=\chi^0=0} = \langle \hat{b}_{\mathrm{out}}(u) \rangle_{\mathrm{DHO}} \\ &+ K \int dt \left(\frac{\delta \langle \left(\bar{\phi}^- \phi^-\right)^2 \rangle [\chi, \chi^0]}{\delta \chi(u)} \bigg|_{\chi=\chi^0=0} - \frac{\delta \langle \left(\bar{\phi}^+ \phi^+\right)^2 \rangle [\chi, \chi^0]}{\delta \chi(u)} \bigg|_{\chi=\chi^0=0} \right) + \mathcal{O}(K^2) \end{aligned} \tag{B.2}$$

We firstly consider the derivatives of the two terms inside the integral of the interaction term in the action with respect to either source field.

$$
\begin{aligned}
&\frac{\delta\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^0]}{\delta\chi(u)}\Bigg|_{\chi=\chi^0=0} \\
&= \frac{\delta}{\delta\chi(u)}\big(2n_B^2 + 3n_B\langle\bar{\phi}\ \ \rangle[\chi]\langle\phi\ \ \rangle[\chi^0] + \langle\bar{\phi}\ \ \rangle^2[\chi]\langle\phi\ \ \rangle^2[\chi^0]\big)\Bigg|_{\chi=\chi^0=0} \\
&= 3n_B\langle\phi\ \ \rangle[0]\,\frac{\delta}{\delta\chi(u)}\langle\bar{\phi}\ \ \rangle[\chi]\Bigg|_{\chi=\chi^0=0} + \langle\phi\ \ \rangle^2[0]\,\frac{\delta}{\delta\chi(u)}\langle\bar{\phi}\ \ \rangle^2[\chi]\Bigg|_{\chi=\chi^0=0}.
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
&\frac{\delta\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^0]}{\delta\chi^0(u)}\Bigg|_{\chi=\chi^0=0} \\
&= \frac{\delta}{\delta\chi^0(u)}\big(2n_B^2 + 3n_B\langle\bar{\phi}\ \ \rangle[\chi]\langle\phi\ \ \rangle[\chi^0] + \langle\bar{\phi}\ \ \rangle^2[\chi]\langle\phi\ \ \rangle^2[\chi^0]\big)\Bigg|_{\chi=\chi^0=0} \\
&= 3n_B\langle\bar{\phi}\ \ \rangle[0]\,\frac{\delta}{\delta\chi^0(u)}\langle\phi\ \ \rangle[\chi]\Bigg|_{\chi=\chi^0=0} + \langle\bar{\phi}\ \ \rangle^2[0]\,\frac{\delta}{\delta\chi^0(u)}\langle\phi\ \ \rangle^2[\chi]\Bigg|_{\chi=\chi^0=0}.
\end{aligned} \tag{B.4}
$$

Now we evaluate the integrand by considering both terms for either source field. We start with the derivatives with respect to χ,

$$
\begin{aligned}
&\frac{\delta\langle\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi(u)}\Bigg|_{\chi=\chi^0=0} = 3n_B\left(-i\sqrt{\kappa}\int dt_1 G^R(t-t_1)f(t_1)\right)\left(\sqrt{\kappa}\frac{F-1}{2}G^R(u-t)\right) \\
&-\left(\kappa\int dt_3\int dt_4 G^R(t-t_3)f(t_3)G^R(t-t_4)f(t_4)\right)\left(i\kappa\int dt_5(F-1)G^R(u-t)G^A(t_5-t)\bar{f}(t_5)\right),
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
&\frac{\delta\langle\big(\bar{\phi}^+(t)\phi^+(t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi(u)}\Bigg|_{\chi=\chi^0=0} = 3n_B\left(-i\sqrt{\kappa}\int dt_1 G^R(t-t_1)f(t_1)\right)\left(\sqrt{\kappa}\frac{F+1}{2}G^R(u-t)\right) \\
&-\left(\kappa\int dt_3\int dt_4 G^R(t-t_3)f(t_3)G^R(t-t_4)f(t_4)\right)\left(i\kappa\int dt_5(F+1)G^R(u-t)G^A(t_5-t)\bar{f}(t_5)\right),
\end{aligned} \tag{B.6}
$$

this leads to the following integrand,

$$
\begin{aligned}
&\frac{\delta\langle\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi(u)}\Bigg|_{\chi=\chi^0=0} - \frac{\delta\langle\big(\bar{\phi}^+(t)\phi^+(t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi(u)}\Bigg|_{\chi=\chi^0=0} \\
&= 3in_B\kappa G^R(u-t)\int dt_1 G^R(t-t_1)f(t_1) \\
&+ 2i\kappa^2 G^R(u-t)\iiint dt_3 dt_4 dt_5 G^R(t-t_3)f(t_3)G^R(t-t_4)f(t_4)G^A(t_5-t)\bar{f}(t_5)
\end{aligned} \tag{B.7}
$$

We turn to evaluating the terms with the derivative with respect to χ^0,

$$\left.\frac{\delta\langle\big(\bar{\phi}^{+}\phi^{+}\big)^2\rangle[\chi,\chi^0]}{\delta\chi^0(u)}\right|_{\chi=\chi^0=0} = 3n_B\left(\int dt_1\sqrt{\kappa}iG^A(t_1-t)\bar{f}(t_1)\right)\left(-\sqrt{\kappa}\frac{F-1}{2}G^A(t-u)\right)$$
$$+\left(-\kappa\iint dt_2dt_3G^A(t_2-t)\bar{f}(t_2)G^A(t_3-t)\bar{f}(t_3)\right)\left(i\kappa(F-1)\int dt_4G^A(t-u)G^R(t_4-t)f(t_4)\right), \tag{B.8}$$

$$\left.\frac{\delta\langle\big(\bar{\phi}\ \ \phi\ \ \big)^2\rangle[\chi,\chi^0]}{\delta\chi^0(u)}\right|_{\chi=\chi^0=0} = 3n_B\left(\sqrt{\kappa}\int dt_1iG^A(t_1-t)\bar{f}(t_1)\right)\left(-\sqrt{\kappa}\frac{F+1}{2}G^A(t-u)\right)$$
$$+\left(-\kappa\iint dt_2dt_3G^A(t_2-t)\bar{f}(t_2)G^A(t_3-t)\bar{f}(t_3)\right)\left(i\kappa(F+1)\int dt_4G^A(t-u)G^R(t-t_4)f(t_4)\right), \tag{B.9}$$

this leads to the following integrand,

$$\left.\frac{\delta\langle\big(\bar{\phi}\ \ (t)\phi\ \ (t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi^0(u)}\right|_{\chi=\chi^0=0} - \left.\frac{\delta\langle\big(\bar{\phi}^{+}(t)\phi^{+}(t)\big)^2\rangle[\chi,\chi^0]}{\delta\chi^0(u)}\right|_{\chi=\chi^0=0}$$
$$= -3in_B\kappa G^A(t-u)\int dt_1G^A(t_1-t)\bar{f}(t_1)$$
$$-2i\kappa^2G^A(t-u)\iiint dt_2dt_3dt_4G^A(t_2-t)\bar{f}(t_2)G^A(t_3-t)\bar{f}(t_3)G^R(t_4-t)f(t_4). \tag{B.10}$$

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