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The thermodynamics of continuous feedback control

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1 Introduction

Quantum thermodynamics is an emerging research field aiming to extend standard thermodynamics and non-equilibrium statistical physics to ensembles of sizes well below the thermodynamic limit, in non-equilibrium situations, and with the full inclusion of quantum effects [1]. In other words, it describes the relationship between two physical theories: thermodynamics and quantum physics.
One subfield of quantum thermodynamics describes open quantum systems with continuous feedback control. A detector measures a quantity of the system, and this measurement result is fed back to the system by changing experimental parameters such as applied electric fields. This alternating process of the detector influencing the system and the system influencing the detector in return leads to a feedback loop between the quantum system and the detector, which both influence each other.
There has been progress in describing these systems in the past three decades. However, these descriptions are typically based on stochastic differential equations, which often have to be solved numerically. Because of this, they provide only limited qualitative insight; for example, their energetics are not yet fully understood [2].
To get more insight into systems with continuous feedback control, the letter "Quantum Fokker-Planck master equation for continuous feedback control" [2] presents a new formalism that allows the derivation of a master equation of the system alone. This change means that instead of a stochastic differential equation which can mostly only be solved numerically, we have an ordinary differential equation that can be used to get insight into the system's behavior.
In this thesis, we will use this new formalism as a starting point to investigate the energetics of systems with continuous feedback control. We do this by comparing the average energy change of a quantum thermodynamic system without continuous feedback control to that of a system with continuous feedback control. After defining heat current and power for the system with continuous feedback control, we take a step back and use stochastic analysis to get further insight into where the terms of heat current and power come from. Since we need stochastic analysis for this step, we provide a small introduction in section Sec. 1.1 to get a rough understanding of the ideas behind the calculations.
Our main results are the calculations of the average energy change in a system with continuous feedback control (23) and the definitions for power and heat current in that system (29), with further insight into how these quantities are influenced by changes in the Hamiltonian and the density matrix of the system.

1.1 Basics of Stochastic Analysis

A one-dimensional differential equation can be written as

$$\frac{dx}{dt} = f(x,t). \tag{1}$$

If we now look at the change dx in an infinitesimal timestep dt instead of the derivative, we can rewrite this equation into the form

$$dx = f(x,t)dt. \tag{2}$$

This form is often called differential form and is closely related to how stochastic differential equations are written down [3]

$$dx = f(x,t)dt + g(x,t)dW. \tag{3}$$

Here the change dx depends on the infinitesimal timestep dt as in an ordinary differential equation. However, now we have this additional term dW, which is added in each time step dt. This additional term is called the Wiener increment and is an infinitesimal Gaussian increment with zero mean and variance dt. This means the change in dx is no longer deterministic; instead, it has a random part. This implies that the variable x is a random variable. Because of this, stochastic differential equations do not have a deterministic function as a solution. Instead, we get a random solution depending on the values of dW in each time step.

This additional factor dW leads to multiple changes in how things are calculated in stochastic analysis. For example, suppose we have a function $y(x)$ dependent on the stochastic variable x we defined above. In that case, the Taylor series expansion in the limit $dt \to 0$ now has to consider second-order terms. This is called Ito's formula [3]

$$dy = \frac{dy}{dx}dx + \frac{1}{2}\frac{d^2y}{dx^2}(dx)^2. \tag{4}$$

Second-order terms do not vanish based on our choice of the variance of the Wiener increment dW. Because it is chosen to be dt this leads to Ito's rule [3]

$$(dW)^2 = dt. \tag{5}$$

Additionally, all terms of the form $dt^n dW^m$, $\forall n, m \geq 1$ vanish in the limit $dt \to 0$. This means that in our example

$$(dx)^2 = g(x,t)^2(dW)^2 = g(x,t)^2 dt. \tag{6}$$

Therefore the last term in (4) is linear in dt and should thus be kept in the limit $dt \to 0$. If x were not a stochastic variable, this term would vanish.

Like standard differential equations, stochastic differential equations can often only be solved numerically. This means they are a vital tool for describing explicit situations well. However, getting a deeper understanding of the processes influencing the solution is complicated when we look at a more general system [2].

2 Heat and work in quantum master equations

Our starting point is a quantum system described by the density matrix $\hat{\rho}$ connected to several baths. This connection means the quantum system can exchange energy in the form of heat with these baths. In this setup, the time evolution of the quantum system is described by a quantum master equation

$$\partial_t \hat{\rho} = -i[\hat{H}(D(t)), \hat{\rho}] + \mathcal{L}_B \hat{\rho} \equiv \mathcal{L}(D(t))\hat{\rho}. \tag{7}$$

The first term of the equation describes the system's unitary time evolution. In contrast the second term describes the non-unitary time evolution produced by the system's energy exchange with the baths [4]. The Hamiltonian $\hat{H}$ is dependent on a function $D(t)$. This function is introduced because we want to compare this setup with a setup with continuous feedback control. Our goal here is to describe the average energy change of the system and see which part of this change contributes to work or heat. We do this both in a deterministic and a stochastic setting.

2.1 Deterministic protocol

In this chapter, we treat the function $D(t)$ as a deterministic function. This leads to the deterministic time evolution of $\hat{\rho}$. This means the average energy change of our system is

$$\partial_t \operatorname{Tr}\{\hat{H}(D(t))\hat{\rho}(t)\} \equiv \partial_t \langle \hat{H}(D(t))\rangle = \partial_t D(t) \langle \partial_D \hat{H}(D(t))\rangle + \operatorname{Tr}\{\hat{H}(D(t))\partial_t \hat{\rho}(t)\}. \tag{8}$$

Thus the average energy change comprises two types of energy transfer. Their intuitive meaning is that of two types of energetic resources, one fully controllable and useful, the other uncontrolled and wasteful.

Generally, an experimenter controls the variation of $\hat{H}$, so the energy change associated with this time variation is associated with work. The uncontrollable energy change is associated with the reconfiguration of the system state and is associated with heat. This means the average power and average heat current are the following [1]

$$\langle P \rangle \equiv \partial_t D(t) \langle \partial_D \hat{H}(D(t))\rangle \qquad \langle J \rangle \equiv \operatorname{Tr}\{\hat{H}(D(t))\partial_t \hat{\rho}\} = \operatorname{Tr}\{\hat{H}(D(t))\mathcal{L}_B \hat{\rho}(t)\}. \tag{9}$$

We used (7) for the heat current to get it into a form we will recover in Sec. 2.2.

The formal definitions of the average heat absorbed by the system after time τ and average work done on the system after time τ is then

$$\langle Q \rangle \equiv \int_0^\tau \partial_t D(t) \langle \partial_D \hat{H}(D(t))\rangle dt \qquad \langle W \rangle \equiv \int_0^\tau \operatorname{Tr}\{\hat{H}(D(t))\mathcal{L}_B \hat{\rho}(t)\} dt. \tag{10}$$

2.2 Stochastic protocol

Instead of a deterministic function, we now change D to a stochastic variable. This means D now follows a general stochastic differential equation

$$dD = A(D)dt + B(D)dW. \tag{11}$$

and from (7) we get $d\hat{\rho}_c = -i[\hat{H}(D(t)), \hat{\rho}_c]dt + \mathcal{L}_B\hat{\rho}_c dt$. Here $\hat{\rho}_c$ is the conditional density matrix which depends on the values of D.

In a stochastic setting, we have to include a third term into the change of the average energy because of the stochastic product rule

$$\partial_t\langle\hat{H}(D)\rangle = \frac{E(\mathrm{Tr}\{d(\hat{H}(D)\hat{\rho}_c)\})}{dt} = \frac{1}{dt}\left(E(\mathrm{Tr}\{d\hat{H}(D)\hat{\rho}_c + \hat{H}(D)d\hat{\rho}_c + d\hat{H}((D)d\hat{\rho}_c\})\right). \tag{12}$$

In this stochastic setting, the average we use differs from the standard quantum mechanical average in Sec. 2. Here, we also average over all outcomes of D denoted by $E(\cdot)$.

We define heat and work in this stochastic protocol similar to the definitions we used in the deterministic protocol. Again the term where we have a variation of $\hat{H}$ is identified as power, and the term where we have a variation of $\hat{\rho}_c$ is identified as heat current

$$\langle P\rangle \equiv \frac{1}{dt}E(\mathrm{Tr}\{d\hat{H}(D(t))\hat{\rho}_c\}) \qquad \langle J\rangle \equiv \frac{1}{dt}E(\mathrm{Tr}\{\hat{H}(D(t))d\hat{\rho}_c\}). \tag{13}$$

In this setting, we have a third term to consider: $\frac{1}{dt}E(\mathrm{Tr}\{d\hat{H}(D(t))d\hat{\rho}_c\})$. This term vanishes here because $d\hat{\rho}_c$ has no direct dependence on the Wiener increment dW. Nevertheless, this term will become relevant when we introduce feedback control. Now we insert $d\hat{\rho}$ and (11) and remember Ito's formula for $d\hat{H}(D)$ (4),

$$d\hat{H}(D) = \partial_D\hat{H}(D)dD + \frac{1}{2}\partial_D^2\hat{H}(D)dD^2. \tag{14}$$

We get the following expression for heat current and power. The whole calculation can be found in App. A.

$$\begin{aligned}\langle P\rangle &= A(D)\left\langle\partial_D\hat{H}(D)\right\rangle + \frac{1}{2}B^2(D)\left\langle\partial_D^2\hat{H}(D)\right\rangle \\ \langle J\rangle &= E(\mathrm{Tr}\{\hat{H}(D)\mathcal{L}_B\hat{\rho}_c\}).\end{aligned} \tag{15}$$

We can see that the heat current does not look different from the heat current in the deterministic protocol. For the power, we now have an additional term with a second derivative that stems from the stochastic setup and a more familiar term with a first derivative of the Hamiltonian as we had before (9). We will use this expression to compare it with the setup with continuous feedback control.

3 Quantum Fokker-Planck master equation

3.1 Continuous feedback control

This section follows the description of continuous feedback control from [2]. Instead of just having a general function $D(t)$, we now continuously measure an arbitrary observable of our quantum system and use the results of that measurement as our function $D(t)$, which in return changes the system.

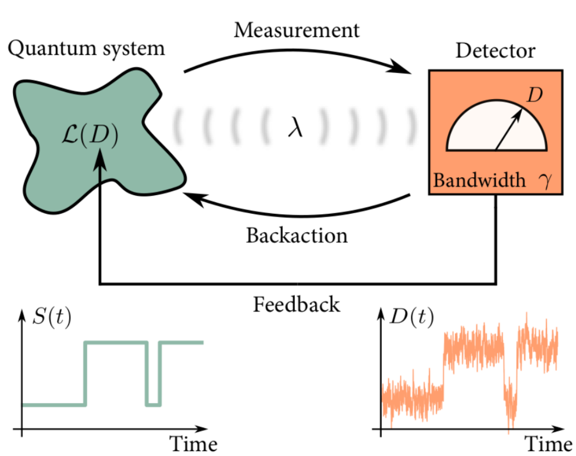

Figure 1: Overview of a measurement and feedback loop between the quantum system and detector. Figure is taken from [2].

We are looking at an open quantum system whose dynamics are described by a Liouville superoperator $\mathcal{L}(D)$ given in (7). Additionally, we are continuously measuring the observable $\hat{A}$ of the system and feeding back this measurement outcome D into the system, which controls the system superoperator $\mathcal{L}(D)$.
The measurement here is not necessarily projective; therefore, the system does not collapse into an eigenstate of the observable $\hat{A}$ after a measurement. However, the measurement still influences the system via a backaction which depends on the measurement strength λ. For $\lambda \to 0$, we have a weak non-intrusive measurement, while for $\lambda \to \infty$, the measurement is strong and projective.
We also consider a finite band-width γ acting as a low pass frequency filter. This eliminates high-frequency measurement noise but introduces a time delay that scales with $1/\gamma$. Because of this, this setup is a realistic description of a detector.

3.2 Quantum Focker-Planck master equations

In this general setup, we can use stochastic calculus to derive a Fokker-Planck master equation for the joint state of the system and detector state $\hat{\rho}_t(D)$. For this, we consider the conditional density matrix, which depends on the random values of the measurements D. This density matrix changes over time as [5]

$$\hat{\rho}_c(t+dt) = e^{\mathcal{L}(D)dt} \frac{\mathcal{M}(z)\hat{\rho}_c(t)}{\mathrm{Tr}\{\mathcal{M}(z)\hat{\rho}_c(t)\}}. \tag{16}$$

Here $\mathcal{M}(z)$ is the measurement superoperator dependent on the unfiltered measurement outcome z at time t. This operator is used to describe the change of the system when measuring outcome z and is dependent on the measurement strength λ [6][7]

$$\mathcal{M}(z)\hat{\rho} \equiv \hat{K}(z)\hat{\rho}\hat{K}^\dagger(z), \qquad \hat{K}(z) = \left(\frac{2\lambda dt}{\pi}\right)^{1/4} e^{-\lambda dt(z-\hat{A})^2}. \tag{17}$$

We can now introduce the following Wiener increment [6], which describes the randomness or noise in the measurement result

$$dW = 2\sqrt{\lambda}dt(z - \langle\hat{A}\rangle_c) \qquad \langle\hat{A}\rangle_c = \mathrm{Tr}\{\hat{A}\hat{\rho}_c(t)\}. \tag{18}$$

Using (18), we get an equation of z, which we can insert into (16). Here we will look at $dt \to 0$; therefore, we can expand (16) to second order in dW and first order in dt, which then results in the Belavkin equation [8]

$$d\hat{\rho}_c = \mathcal{L}(D)\hat{\rho}_c dt + \lambda\mathcal{D}[A]\hat{\rho}_c dt + \sqrt{\lambda}\{\hat{A} - \langle\hat{A}\rangle_c, \hat{\rho}_c\}dW \qquad \mathcal{D}[\hat{A}]\hat{\rho} = \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A}, \hat{\rho}\}. \tag{19}$$

Additionally, the relation between unfiltered and filtered measurement outcomes is given by the following equation. Remember that γ is the detector bandwidth which is introduced through a low-pass frequency filter [9]

$$D(t) = \int_{-\infty}^{t} ds\gamma e^{-\gamma(t-s)}z(s) \implies dD = \gamma(z - D)dt = \gamma(\langle\hat{A}\rangle_c - D)dt + \frac{\gamma}{2\sqrt{\lambda}}dW. \tag{20}$$

With this preparation, we can derive a Fokker-Planck master equation for $\hat{\rho}_t$ using the stochastic product and chain rules. The complete calculation is described in [2]

$$\partial_t\hat{\rho}_t(D) = \mathcal{L}(D)\hat{\rho}_t(D) + \lambda\mathcal{D}[\hat{A}]\hat{\rho}_t(D) - \gamma\partial_D\mathcal{A}(D)\hat{\rho}_t(D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{\rho}_t(D) \qquad \mathcal{A}(D)\hat{\rho} = \frac{1}{2}\{\hat{A} - D, \hat{\rho}\}. \tag{21}$$

The first term describes the system's evolution, the second describes how the system dephases in the eigenbasis of $\hat{A}$ proportional to λ due to measurement backaction, and the last two terms describe the evolution of the detector.

4 Heat & work in Quantum Focker-Planck master equations

4.1 Averages

Similar to how we described heat and work in general quantum thermodynamic systems in Sec. 2, we now try to find expressions for heat and work in quantum systems with continuous feedback control.
To this end, we start with the average energy change of our system $\partial_t\langle\hat{H}(D)\rangle \equiv \partial_t \,\mathrm{Tr}\left\{\int dD\hat{H}(D)\hat{\rho}_t(D)\right\}$. As in Sec. 2.2, we take not just the quantum mechanics average for the average energy change but also the average over all measurement outcomes D.
We can use (21) to rewrite the average energy change. After shifting the derivative inside the trace, this leaves us with the following expression

$$\partial_t\langle\hat{H}(D)\rangle = \mathrm{Tr}\left\{\int dD\hat{H}(D)\left[\mathcal{L}(D)\hat{\rho}_t(D) + \lambda\mathcal{D}[\hat{A}]\hat{\rho}_t(D) - \gamma\partial_D\mathcal{A}(D)\hat{\rho}_t(D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{\rho}_t(D)\right]\right\}. \tag{22}$$

Using partial integration and the fact that the trace is cyclic $\mathrm{Tr}\{\hat\rho_1\hat\rho_2\hat\rho_3\} = \mathrm{Tr}\{\hat\rho_2\hat\rho_3\hat\rho_1\}$, we can transform this equation into the form

$$\partial_t\langle\hat H(D)\rangle = \mathrm{Tr}\left\{\int dD\hat H(D)\mathcal{L}(D)\hat\rho_t(D)\right\} + \left\langle \lambda\mathcal{D}[\hat A]\hat H(D) + \gamma\mathcal{A}(D)\partial_D\hat H(D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat H(D)\right\rangle. \tag{23}$$

The entire calculation can be found in App. B. We can define heat and work in a similar way we did before

$$\langle P\rangle = \gamma\left\langle \mathcal{A}(D)\partial_D\hat H(D)\right\rangle + \frac{\gamma^2}{8\lambda}\left\langle\partial_D^2\hat H(D)\right\rangle, \tag{24}$$

$$\langle J\rangle = \mathrm{Tr}\left\{\int dD\hat H(D)\mathcal{L}_B\hat\rho_t(D)\right\} + \lambda\left\langle\mathcal{D}[\hat A]\hat H(D)\right\rangle. \tag{25}$$

Comparing this definition to how we defined heat and power in (15), we can see that three of the four terms are familiar. Instead of a prefactor in front of the derivatives of $\hat H$, we now have superoperators acting on them.
The only completely new term is the second term of the heat current. It shows up because we are now looking at a system with feedback control and, therefore, with a measurement backaction that changes the system state with each measurement. This term is sometimes called "quantum heat" [10].

4.2 Stochastic heat & work

In this section, to understand better where the different terms in the power and heat current come from, we take a step back and look at the energy changes using the Belavkin equation (19). Therefore we treat our measurement variable D as a stochastic variable to recover a similar result as in section 4.1 but get further insight into where the terms in (23) come from. Using stochastic variables, the average energy change can be written as

$$\partial_t\langle\hat H(D)\rangle = \frac{E(\mathrm{Tr}\{d(\hat H\hat\rho_c)\})}{dt}. \tag{26}$$

Here $\hat\rho_c$ is the conditional density matrix we introduced in section 3.2. D is a stochastic variable that leads to $\hat\rho_c$ and $\hat H(D)$, also being stochastic variables. $d\hat\rho_c$ and D follow the stochastic differential equations (19) and (20). For $d\hat H(D)$, we can use Ito's formula (14).
As in chapter 4.2, using the stochastic product rule, we see that three terms influence system's energy change

$$\frac{E(\mathrm{Tr}\{d(\hat H\hat\rho_c)\})}{dt} = \frac{1}{dt}\left(E(\mathrm{Tr}\{d\hat H(D)\hat\rho_c)\} + E(\mathrm{Tr}\{\hat H(D)d\hat\rho_c)\} + E(\mathrm{Tr}\{d\hat H(D)d\hat\rho_c)\}\right). \tag{27}$$

Compared to chapter 2.2, here, the third term $d\hat H(D)d\hat\rho_c$ does not vanish since $d\hat\rho_c$ has a direct dependence on the Wiener increment dW.
We can now replace the stochastic variables from (27) to transform each term independently. The complete calculations of this step can be found in section App. C.

$$\begin{aligned}
\frac{1}{dt}(E(\mathrm{Tr}\{d\hat H(D)\hat\rho_c\})) &= \gamma\langle\hat A\rangle_c E(\mathrm{Tr}\{\partial_D\hat H(D)\hat\rho_c\}) - \gamma E(\mathrm{Tr}\{D\partial_D\hat H(D)\hat\rho_c\}) + \frac{\gamma^2}{8\lambda}E(\mathrm{Tr}\{\partial_D^2\hat H(D)\hat\rho_c\})\\
\frac{1}{dt}(E(\mathrm{Tr}\{\hat H(D)d\hat\rho_c\})) &= E(\mathrm{Tr}\{\hat H(D)\mathcal{L}_B\hat\rho_c\}) + \lambda E(\mathrm{Tr}\{\mathcal{D}[\hat A]\hat H(D)\hat\rho_c\})\\
\frac{1}{dt}(E(\mathrm{Tr}\{d\hat H(D)d\hat\rho_c\})) &= \frac{\gamma}{2}E(\mathrm{Tr}\{\partial_D\hat H(D)\hat A\hat\rho_c\}) + \frac{\gamma}{2}E(\mathrm{Tr}\{\hat A\partial_D\hat H(D)\hat\rho_c\}) - \gamma\langle\hat A\rangle_c E(\mathrm{Tr}\{\partial_D\hat H(D)\hat\rho_c\}).
\end{aligned} \tag{28}$$

We define heat current and power in the following way to recover the same definitions for them as in section 4.1. Notice that for the power two terms from the first and third line from (28) do cancel out. The full calculations of this step can also be found in section App. C.

$$\begin{aligned}
\langle P\rangle &\equiv \frac{1}{dt}E(\mathrm{Tr}\{d\hat H(D(t))\hat\rho_c + d\hat H(D)d\hat\rho_c\}) = E(\mathrm{Tr}\{\frac{\gamma}{2}\left(\{\partial_D\hat H(D),\hat A\} - 2D\partial_D\hat H(D)\right)\hat\rho_c + \frac{\gamma^2}{8\lambda}\partial_D^2\hat H(D)\hat\rho_c\})\\
&= \gamma\left\langle\mathcal{A}(D)\partial_D\hat H(D)\right\rangle + \frac{\gamma^2}{8\lambda}\left\langle\partial_D^2\hat H(D)\right\rangle\\
\langle J\rangle &\equiv \frac{1}{dt}E(\mathrm{Tr}\{\hat H(D(t))d\hat\rho_c\}) = E(\mathrm{Tr}\{\hat H(D)\mathcal{L}(D) + \lambda\mathcal{D}[\hat A]\hat H(D)\})\\
&= E(\mathrm{Tr}\{\hat H(D)\mathcal{L}_B\}) + \lambda\left\langle\mathcal{D}[\hat A]\hat H(D)\right\rangle.
\end{aligned} \tag{29}$$

Our definition for the heat current matches the definition we used in section 2.2; the only difference is that now we have a quantum heat term. Thus when defining heat as the fully uncontrollable part of the average energy change, we get to the definition of heat current in (25).

We learn from the stochastic calculation that the new term $d\hat{H}d\hat{\rho}_c$, which shows up because we now use feedback control is part of the definition of power from the last section (25). The way we defined power before, back in chapter 2.1, was that we associated the energy change that is controllable with power. Here in our definition of power with feedback control, the term $d\hat{H}d\hat{\rho}_c$ is not fully controllable because it also depends on the change in the state of the system, which involves randomness based on the measurement results. Therefore it might make sense to call this term by a different name. One possibility would be to call it stochastic power to point out the inherent randomness of the measurement outcome that influences this term.

5 Conclusion

Based on the new formalism outlined in [2], we did derive the average energy change of a system with continuous feedback control (23). By comparing the terms of this result to the terms for power and heat current in the average energy change of a quantum thermodynamic system without feedback control (8), we defined power and heat current for the system with feedback (25).

After defining heat current and power, we used stochastic analysis to get further qualitative insight. In the case of a system with continuous feedback control, the power is also dependent on a not fully controllable term that shows up because of the inherent randomness of the measurement result. Instead, the power is dependent on a term that is both influenced by changes in the Hamiltonian and changes in the density matrix of the state. (29)

The next step to understanding this term would be to look at explicit examples of systems to see if it behaves like power or if it is more fitting to call it by a different name.

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A Details to the calculations of section 2.2

In this section we do the calculation to get to equation (15). We insert $d\hat{\rho}_c$ and $d\hat{H}(D)$ into (13) to get the following

$$\begin{aligned}\langle P\rangle &= \frac{1}{dt}\left(E\left[\mathrm{Tr}\{(\partial_D\hat{H}(D)dD+\frac{1}{2}\partial_D^2\hat{H}(D)dD^2)\hat{\rho}_c\right]\right)\\ \langle J\rangle &= \frac{1}{dt}E(\mathrm{Tr}\{\hat{H}(D)(-i[\hat{H}(D),\hat{\rho}_c]dt+\mathcal{L}_B\hat{\rho}_c dt)\}).\end{aligned} \tag{30}$$

Next we insert (11) and use that the trace is cyclic to get

$$\begin{aligned}\langle P\rangle &= \frac{1}{dt}\left(E\left[\mathrm{Tr}\{(\partial_D\hat{H}(D)A(D)\hat{\rho}_c dt+\underbrace{\partial_D\hat{H}(D)B(D)\hat{\rho}_c dW}_{=0}+\frac{1}{2}\partial_D^2\hat{H}(D)B^2(D)\hat{\rho}_c dt\right]\right)\\ \langle J\rangle &= \frac{1}{dt}E(\mathrm{Tr}\{\underbrace{-i\hat{H}(D)[\hat{H}(D),\hat{\rho}_c]dt}_{=0}+\hat{H}\mathcal{L}_B\hat{\rho}_c dt)\}).\end{aligned} \tag{31}$$

Now with some rearranging we get to (15).

B Details to the calculations of section 4.1

In this section we derive (23) based on (22). In the following we transform each of the summands seperately.

$$\begin{aligned}\lambda\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\mathcal{D}[\hat{A}]\hat{\rho}_t(D)\right\} &= \lambda\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\left(\hat{A}\hat{\rho}_t(D)\hat{A}^\dagger-\frac{1}{2}\hat{A}^\dagger\hat{A}\hat{\rho}_t(D)-\frac{1}{2}\hat{\rho}_t(D)\hat{A}^\dagger\hat{A}\right)\right\}\\ &= \lambda\,\mathrm{Tr}\left\{\int dD\,\underbrace{\hat{H}(D)\hat{A}\hat{\rho}_t(D)\hat{A}^\dagger}_{\hat{A}\hat{H}(D)\hat{A}^\dagger\hat{\rho}_t(D)}-\frac{1}{2}\hat{H}(D)\hat{A}^\dagger\hat{A}\hat{\rho}_t(D)-\frac{1}{2}\underbrace{\hat{H}(D)\hat{\rho}_t(D)\hat{A}^\dagger\hat{A}}_{\hat{A}^\dagger\hat{A}\hat{H}(D)\hat{\rho}_t(D)}\right\}\\ &= \lambda\,\mathrm{Tr}\left\{\int dD\hat{A}\hat{H}(D)\hat{A}^\dagger\hat{\rho}_t(D)-\frac{1}{2}\hat{H}(D)\hat{A}^\dagger\hat{A}\hat{\rho}_t(D)-\frac{1}{2}\hat{A}^\dagger\hat{A}\hat{H}(D)\hat{\rho}_t(D)\right\}\\ &= \lambda\,\mathrm{Tr}\left\{\int dD\mathcal{D}[\hat{A}]\hat{H}(D)\hat{\rho}_t(D)\right\} = \lambda\langle\mathcal{D}[\hat{A}]\hat{H}(D)\rangle.\end{aligned} \tag{32}$$

$$\begin{aligned}-\gamma\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\partial_D\mathcal{A}(D)\hat{\rho}_t(D)\right\} &= -\gamma\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\partial_D\left(\frac{1}{2}(\hat{A}-D)\hat{\rho}_t(D)+\hat{\rho}_t(D)\frac{1}{2}(\hat{A}-D)\right)\right\}\\ &= -\gamma\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\partial_D\left(\frac{1}{2}\hat{A}\hat{\rho}_t(D)+\frac{1}{2}\hat{\rho}_t(D)\hat{A}-D\hat{\rho}_t(D)\right)\right\}\\ &= \gamma\,\mathrm{Tr}\left\{\int dD\,\frac{1}{2}\partial_D\hat{H}(D)\hat{A}\hat{\rho}_t(D)+\frac{1}{2}\underbrace{\partial_D\hat{H}(D)\hat{\rho}_t(D)\hat{A}}_{\hat{A}\partial_D\hat{H}(D)\hat{\rho}_t(D)}-\partial_D\hat{H}(D)D\hat{\rho}_t(D)\right\}\\ &= \gamma\,\mathrm{Tr}\left\{\int dD\left(\frac{1}{2}\partial_D\hat{H}(D)\hat{A}+\frac{1}{2}\hat{A}\partial_D\hat{H}(D)-\partial_D\hat{H}(D)D\right)\hat{\rho}_t(D)\right\}\\ &= \gamma\,\mathrm{Tr}\left\{\int dD\mathcal{A}(D)\partial_D\hat{H}(D)\hat{\rho}_t(D)\right\} = \gamma\langle\mathcal{A}(D)\partial_D\hat{H}(D)\rangle.\end{aligned} \tag{33}$$

$$\frac{\gamma^2}{8\lambda}\,\mathrm{Tr}\left\{\int dD\hat{H}(D)\partial_D^2\hat{\rho}_t(D)\right\} = \frac{\gamma^2}{8\lambda}\,\mathrm{Tr}\left\{\int dD\partial_D^2\hat{H}(D)\hat{\rho}_t(D)\right\} = \frac{\gamma^2}{8\lambda}\langle\partial_D^2\hat{H}(D)\rangle. \tag{34}$$

This means our expression for the change of energy of our system transforms into the following expression. This expression is equal to (23)

$$\mathrm{Tr}\left\{\int dD\left(\hat{H}(D)\mathcal{L}(D)+\lambda\mathcal{D}[\hat{A}]\hat{H}(D)+\gamma\mathcal{A}(D)\partial_D\hat{H}(D)+\frac{\gamma^2}{8\lambda}\partial_D^2\hat{H}(D)\right)\hat{\rho}_t(D)\right\}. \tag{35}$$

C Details to the calculations of section 4.2

In this chapter we derive (28) using stochastic analysis.

$$
\begin{aligned}
\frac{1}{dt}E(\mathrm{Tr}\{d\hat{H}(D)\hat{\rho}_c\} &= \frac{1}{dt}E(\mathrm{Tr}\{\partial_D\hat{H}(D)\hat{\rho}_c dD + \frac{1}{2}\partial_D^2\hat{H}(D)\hat{\rho}_c dD^2\}) \\
&= \frac{1}{dt}E(\mathrm{Tr}\{\partial_D\hat{H}(D)\hat{\rho}_c\left(\gamma(\langle\hat{A}\rangle_c - D)dt + \frac{\gamma}{2\sqrt{\lambda}}dW\right) + \frac{1}{2}\partial_D^2\hat{H}(D)\hat{\rho}_c\frac{\gamma^2}{4\lambda}dt\}) \\
&= E(\mathrm{Tr}\{\partial_D\hat{H}(D)\hat{\rho}_c\gamma(\langle\hat{A}\rangle_c - D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{H}(D)\hat{\rho}_c\}) \\
&= E(\mathrm{Tr}\{\gamma\langle\hat{A}\rangle_c\partial_D\hat{H}(D)\hat{\rho}_c - \gamma D\partial_D\hat{H}(D)\hat{\rho}_c + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{H}(D)\hat{\rho}_c\}).
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\frac{1}{dt}E(\mathrm{Tr}\{\hat{H}(D)d\hat{\rho}_c\} &= \frac{1}{dt}E(\mathrm{Tr}\{\hat{H}(D)\mathcal{L}(D)\hat{\rho}_c dt + \hat{H}(D)\lambda\mathcal{D}[\hat{A}]\hat{\rho}_c dt + \hat{H}(D)\sqrt{\lambda}\{\hat{A} - \langle\hat{A}\rangle_c, \hat{\rho}_c\}dW\}) \\
&= E(\mathrm{Tr}\{\hat{H}(D)\mathcal{L}(D)\hat{\rho}_c + \gamma\left[\underbrace{\hat{H}(D)\hat{A}\hat{\rho}_c\hat{A}}_{\hat{A}\hat{H}(D)\hat{A}\hat{\rho}_c} - \frac{1}{2}\hat{H}(D)\hat{A}\hat{A}\hat{\rho}_c - \frac{1}{2}\underbrace{\hat{H}(D)\hat{\rho}_c\hat{A}\hat{A}}_{\hat{A}\hat{A}\hat{H}(D)\hat{\rho}_c}\right]\}) \\
&= E(\mathrm{Tr}\{\hat{H}(D)\mathcal{L}(D)\hat{\rho}_c + \lambda\mathcal{D}[\hat{A}]\hat{H}(D)\hat{\rho}_c\}).
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\frac{1}{dt}E(\mathrm{Tr}\{d\hat{H}(D)d\hat{\rho}_c\} &= \frac{1}{dt}E(\mathrm{Tr}\{\left(\partial_D\hat{H}(D)dD + \frac{1}{2}\partial_D^2\hat{H}(D)dD^2\right)\left[\mathcal{L}(D)\hat{\rho}_c dt + \lambda\mathcal{D}[\hat{A}]\hat{\rho}_c dt + \sqrt{\lambda}\{\hat{A} - \langle\hat{A}\rangle_c, \hat{\rho}_c\}dW\right]\}) \\
&= E(\mathrm{Tr}\{\partial_D\hat{H}(D)\frac{\gamma}{2}\{\hat{A} - \langle\hat{A}\rangle_c, \hat{\rho}_c\}\}) \\
&= E(\mathrm{Tr}\{\partial_D\hat{H}(D)\frac{\gamma}{2}\left(\hat{A}\hat{\rho}_c + \hat{\rho}_c\hat{A} - 2\langle\hat{A}\rangle_c\hat{\rho}_c\right)\}) \qquad (38) \\
&= E(\mathrm{Tr}\{\frac{\gamma}{2}\left(\partial_D\hat{H}(D)\hat{A}\hat{\rho}_c + \underbrace{\partial_D\hat{H}(D)\hat{\rho}_c\hat{A}}_{\hat{A}\partial_D\hat{H}(D)\hat{\rho}_c} - 2\partial_D\hat{H}(D)\langle\hat{A}\rangle_c\hat{\rho}_c\right)\}). \qquad (39)
\end{aligned}
$$

For the third part of the calculation remember that any multiplication of dt and dW vanishes. Therefore the only product that survives is the one where we multiply $dW \cdot dW = dt$.
If we want to recover (23), we can combine (36) and (38); two terms cancel each other out, and we are left with the following expressions

$$
\begin{aligned}
&= E(\mathrm{Tr}\{-\gamma D\partial_D\hat{H}(D)\hat{\rho}_c + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{H}(D)\hat{\rho}_c + \frac{\gamma}{2}\left(\partial_D\hat{H}(D)\hat{A}\hat{\rho}_c + \hat{A}\partial_D\hat{H}(D)\hat{\rho}_c\right)\}) \\
&= E(\mathrm{Tr}\{\gamma\mathcal{A}(D)\partial_D\hat{H}(D) + \frac{\gamma^2}{8\lambda}\partial_D^2\hat{H}(D)\hat{\rho}_c\}).
\end{aligned} \tag{40}
$$

Combining this expression with (37) we get to (23).